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Allocation of resources to maximize power in analysis of covariance when both variables are measured with error

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Boonplook Chaiket

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INTRODUCTION

In the development of psychology, Thorndike and Hagen (1961) wrote:

> Psychology in 1850 was still in large measure a part of philosophy .... Psychology was almost entirely non-experimental.
>
> By 1900 psychology had felt the impact of the physical and biological sciences and was striving mightily to become a science itself. It was shaking off the ties that bound it to philosophy and forming new alliances with the biological sciences. It had adopted the experimental method and was measurement-conscious. The basic tool of experimentation is measurement, and psychology was expanding its measurement techniques in all directions (pp. 2-3).

Thus far, however, most psychological variables or constructs, such as individual abilities, attitudes, or emotions, are not so precisely measured as biological or physical variables, such as weight or height. Therefore, reliability of measurement or measurement error (ME) has been a main concern within scientific psychology.

Statistical methods and techniques are important tools of experimental research. Research data from any measurements or observations are usually analyzed using statistical techniques to arrive at interpretation of results and conclusions. Most statistical techniques, however, are developed by applied statisticians in the fields of biology and agriculture. They are then optimally applicable to research in those fields. Researchers in other fields, for example in psychology, have to be very careful in using these techniques. This is true also for the analysis of covariance (ANOCO) technique, the focus of this dissertation, that was introduced by Fisher (1934), an eminent applied statistician in biology and agriculture. He introduced ANOCO for the purpose of securing more precision than analysis of variance (ANOVA) in testing for treatment differences. As Fisher (1934) stated: Analysis of covariance technique ... combines the advantages and reconciles the requirement of the two widely applicable procedures known as regression and analysis of variance. Fisher's ANOCO classical model, called here the true score model (TSM) is as follows:

$$y_{ij} = \mu + \alpha_i + \beta x_{ij} + \eta_{ij} \; ; \qquad \begin{array}{l} i = 1,2,\ldots,k \\ j = 1,2,\ldots,n \end{array} \qquad (1)$$

ANOCO is a valid technique if we assume the following:

1a) x is a fixed variable measured without error.

1b) $\eta_{ij} \sim NID(0,\sigma_\eta^2)$ where η_{ij} is sampling error. This means that y scores at a given x have a normal distribution with equal variance, i.e., the homocedasticity assumption.

1c) Subjects are randomly selected from a defined population or the subjects within a treatment are selected randomly from different populations.

1d) There is no slope by treatment interaction, i.e., the slopes are the same for different treatments.

1e) Also, Equation (1) implies, within each treatment, y scores have a linear regression on x.

More detail and information on TSM of ANOCO in terms of its uses, natures and advantages can be found in Biometrics, Vol. 13, 1957 with eminent contributors such as Cochran. Researchers in education and psychology will find the discussion by Elashoff (1969) and Lindquist (1953) of more interest. However, more discussion on the robustness and consequences of such failures to meet the above assumptions are in those of Antigullah (1964), Evans and Anastasio (1968), Lord (1960, 1962, 1967) as well as Glass, Peckham and Saunders (1972). Of these assumptions, 1c is the most important for psychologists since the combination of measurement error (violation of assumption 1a) with differences between groups with respect to the covariate, leads to spurious results. Discussion of this problem is presented next.

As discussed by many educators and psychologists (cf. Elashoff, 1969; Lord, 1960, 1962, 1967) as well as statisticians (cf. Antigullah, 1964; Cochran, 1968), this TSM is not appropriate to use in educational and psychological research and in other research where measurement error is of importance. This is because of the failure of those variables such as psychological variables to meet the assumption of perfect measurement or measurement without error. Assumption in classical measurement theory in psychology (cf. Gulliksen, 1950) states that any measures from testing situations or observations in natural situations are subject to error or are fallible. The repeated measure or observation of the same psychological variables or constructs is considered to vary around the true parts of the variables. That is,

$$X = x + \varepsilon, \quad \text{and}$$

$$Y = y + e, \quad \text{where}$$

X and Y are observable scores, x and y are the latent or true parts of X and Y, respectively. The ME parts of X and Y are denoted by ε and e, respectively. Psychologists working with measurement theory assume also the normality and independence of these ME's with their true scores. That is

$$\varepsilon \sim NID(0, \sigma_\varepsilon^2)$$

and

$$e \sim NID(0, \sigma_e^2) \quad .$$

The definition of reliability of any observable score is the ratio of the variance of the true part to the variance of the true part plus the variance of error. Therefore, the reliabilities of observed scores, X and Y are

$r_{XX} = \dfrac{\sigma_x^2}{\sigma_x^2 + \sigma_\varepsilon^2}$ and $r_{YY} = \dfrac{\sigma_y^2}{\sigma_y^2 + \sigma_e^2}$, respectively (cf. Gulliksen, 1950; Lord and Novick, 1968; Nunnally, 1967).

Consider the case where X and Y are two tests. The reliabilities of tests X and Y can be enhanced by repeated measures or increasing test lengths. For example, if test Y is lengthened by a factor $\underline{a}$ the reliability of the lengthened test, $r_{YYa} = \dfrac{\sigma_y^2}{\sigma_y^2 + \sigma_{e/a}^2} = \dfrac{ar_{YY}}{1+(a-1)r_{YY}}$ (Ebel, 1972). It can be seen from this formula that the factor $\underline{a}$ influences the reliability by multiplying the ME variance by $\frac{1}{a}$.

Psychologists and test theorists have been working with the problem of reliability and its estimation. Many methods of estimation have been proposed; Cronbach's α (Cronbach, 1951); KR-20 and KR-21 (Kuder and Richardson, 1937) as well as split half and parallel tests from the domain sampling (cf. Nunnally, 1967). Thorough discussion of these estimates and their properties can be found in Lord and Novick (1968). From Lord and Novick's (1968) discussion we find that these estimates are biased. However, knowing the reliability of any test, for example of Y, $\hat{r}_{YY}$, and asymptotically unbiased estimate of ME variance can be obtained by employing the following relationship:

$$\hat{\sigma}_e^2 = \hat{\sigma}_Y^2(1-\hat{r}_{YY}) \quad .$$

The relationship between two variables depends on the reliabilities of each of the variables. This is because the true correlation between X and Y, if measurement without error occurs, is $\hat{r}_{xy} = \dfrac{\hat{r}_{XY}}{(\hat{r}_{XX}\hat{r}_{YY})^{1/2}}$. Psychologists call this, the correction for attenuation. When two tests are lengthened, i.e., there are changes in reliabilities, the observed correlation is changed from $\hat{r}_{XY}$ to

$$\hat{r}_{X_bY_a} = \frac{\hat{r}_{XY}\hat{r}_{XX_b}^{1/2}\hat{r}_{YY_a}^{1/2}}{\hat{r}_{XX}^{1/2}\hat{r}_{YY}^{1/2}}$$ where a and b are the factors by which tests Y and X, respectively, are lengthened. More details on the topic of measurement theory can be found in most psychometrics books (cf. Cronbach, 1960; Guilford, 1954; Gulliksen, 1950; Lord and Novick, 1968).

Ordinarily, this important concept of ME or reliability is ignored in the application of most statistical methods. All of the variables or observable scores have been treated as if they were errorless. The source of error is focused only on sampling error, η_{ij} rather than ME's, ε_{ij} and e_{ij} .

In this dissertation ME is brought into ANOCO. The ANOCO which is appropriate to use in educational and psychological research when ME is an important variable, is the theme of this effort.

Related Literature

Modified model of ANOCO

Dissatisfaction with the classical TSM of ANOCO has led many statistically oriented psychologists and educators to modify it and/or search for a new approach. Such a new approach might be more appropriate to be used by researchers in psychology and education than current methods. The assumption in TSM that educational or psychological variables or traits are measured without error is seldom approximated. Lord (1960), a psychometrician, may have been the first who included ME in ANOCO. In 1962, Lord said:

> Making allowance for initial differences among groups on a poor measure of some variables is not the same as making allowance for initial differences on the variables itself. If the variable in question cannot be reliably measured it should be controlled experimentally (by randomization) if possible. Otherwise, some special modification of analysis of covariance is desirable (Lord, 1960) . . . .

Porter (1967) and DeGracie (1968) followed and extended Lord's (1960) modification. This modified model, called here an observed score model (OSM) is as follows:

$$Y_{ij} = \mu + \alpha_i + \beta X_{ij} + \eta_{ij} \; ; \quad i = 1,2,\ldots,k \quad j = 1,2,\ldots,n \quad . \tag{2}$$

The following assumptions from measurement theory were included in place of the assumption of the variable measured without error:

$$2a) \quad Y_{ij} = y_{ij} + e_{ij} \quad ,$$

$$2b) \quad X_{ij} = x_{ij} + \varepsilon_{ij} \quad ,$$

$$2c) \quad e_{ij} \sim NID(0,\sigma_e^2) \quad ,$$

and

$$2d) \quad \varepsilon_{ij} \sim NID(0,\sigma_\varepsilon^2) \quad .$$

Employing this OSM when observations occur in groups non-randomly, Lord (1960), Porter (1967) as well as DeGracie (1968) provided statistics to test unbiased estimates of treatment differences. Their statistics come from unbiased estimate of regression coefficient and result in more appropriate tests than methods derived from the TSM.

Effect of ME on ANOCO

The combination of the effects of ME in regression and ANOVA is the effect of ME on ANOCO. In regression analysis X is assumed to be measured without error. Whenever, the observed score X is available instead of the true score x, the least square estimate of the regression coefficient is not appropriate. In a paper by Berkson (1950), it was demonstrated that the estimate of regression coefficient β', defined by the fallibly measured Y and X, is equal to the regression coefficient β defined by the true parts of Y and X multiplied by the reliability of X. In our notation

$$\beta' = \beta\hat{r}_{XX} = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_\varepsilon^2}\,\beta \quad .$$

That is, $\hat{r}_{XX}$ or $\dfrac{\sigma_X^2}{\sigma_X^2 + \sigma_\varepsilon^2}$ defines the bias of the least square estimate of β. In other words, the estimate of β in a fallible variable is biased by approximately the quantity $1-\hat{r}_{XX}$ (cf. Cochran, 1970; Cleary, Linn and Walster, 1970).

Sutcliffe (1958) may have been the first to investigate the effect of ME on ANOVA when Y is subject to ME. His result indicates that with other things being equal, the power of an F-test of treatment differences decreased as ME in Y is introduced. The effect of ME in Y is to decrease precision and thus increase the probability of type II error, which is a decrease in the power of the test. The decrease in precision can be seen by the fact that now $\sigma_Y^2 = \sigma_y^2 + \sigma_e^2$ (cf. Cleary and Linn, 1969; Porter, 1971).

Literature directed toward the effect of ME on ANOCO is sparse. However, in combining the effect of ME on ANOVA and regression analysis, ME does not only decrease precision but may also lead to a false conclusion in ANOCO. That is, by neglecting ME in ANOCO, it can result in a false conclusion of a non-zero treatment effect when the actual treatment effects are zero (type I error), or the real treatment differences cannot be detected (type II error). This was discussed by Lord (1960), Porter (1971) and also Werts and Linn (1971) for the case of quasi-experimental studies. However, if observations occur in treatment groups through a random process, then errors in X and Y result in only the loss of power, while the test statistic remains appropriate.

Allocation of resources

Relevant literature shows that more powerful tests can be obtained through varying sources of variation. In other words, within an available resource some allocations will yield more a powerful test than others. The relevant literature is by Cronbach and Gleser (1965), Cleary and Linn (1969), and Overall and Dalal (1965). In using selection tests for predictive purposes, Cronbach and Gleser (1965) stated their cost model as follows:

$$C_T = n(C_0 + rC_1) \tag{3}$$

where C_T = total cost or total available resources

C_0 = cost/subject, assumed to be fixed

C_1 = cost/test unit, assumed to be the same for each of all r units of repeated measures, and

n = number of subjects or sample size.

Based on the above model, Overall and Dalal (1965) investigated an allocation strategy to gain more powerful tests in ANOVA. They varied $\underline{n}$ and $\underline{r}$ within the available resources. Their results indicate that the largest $\underline{n}$ with $\underline{r} = 1$ yield the most powerful F-test. However, it was shown by Cleary and Linn (1969) that the above conclusion is true only under the condition of zero fixed cost, i.e., $C_0 = 0$. Otherwise, the maximum power is not obtained at the largest possible sample size with one unit of measure. Cleary and Linn (1969) empirically demonstrated that with the larger sized fixed cost C_0, the maximum power of F-test in ANOVA is at larger $\underline{r}$ and smaller $\underline{n}$ than those in Overall and Dalal (1965). One implication of Cleary and Linn (1969) result is that for some C_0, a combination of size of $\underline{r}$ and $\underline{n}$ besides $\underline{r} = 1$ will result in the most powerful test.

Despite intensive studies in the problem of ME and only a few studies of allocation of resource to maximize power, no study has been done within the framework of ANOCO. The concept of the cost model suggests an approach to gain a more powerful F-test in ANOCO by employing an allocation strategy. This is the purpose of this dissertation, to set up a strategy in allocation of available resources such that the maximum power of the F-test of OSM of ANOCO is obtained. Powers of tests are to be compared while their degrees of freedom (dfs) are held constant. This is done by applying an allocation strategy to X and Y after setting aside the fixed cost for a specified sample size. In order to avoid the problem of the artifact that occurs as a result of attempting to adjust for differences in the variate Y occurring as a result of differences in the covariate X when the covariate is fallibly measured, this dissertation is concerned only with the case where observational units occur in groups randomly.

In the Cleary and Linn (1969) study, allocation of resources to either measurement or sample size was considered and, as in most problems of this type, asymptotic results were derived. However, for small sample sizes the value of an F or $\underline{t}$ statistic depends on the sample size. As a result these asymptotic results will allocate too many resources to measurement as opposed to taking larger samples. In this study, sample size is not a variable considered in the allocation problem so that the asymptotic results herein might be expected to be applicable even for small samples.

This dissertation starts with the development of the allocation formula for X and Y within any available measurement resource. This development appears in section 2. After the development of the allocation formula, a Monte Carlo investigation of this formula and its ramifications follows. Finally, the data from one experiment are used as an example of the application of the strategy. A general discussion concludes this dissertation.

DEVELOPMENT OF THE FORMULA OF ALLOCATION OF AVAILABLE RESOURCES IN ANOCO

The development was based on the modified ANOCO model or OSM. Using this model, the Cronbach and Gleser's (1965) cost model can be extended to

$$C_T = N(C_0 + a_1 C_X + a_2 C_Y) \tag{4}$$

where C_T = total cost or time available,

C_0 = cost or time used by each subject assumed to be fixed,

C_X = cost or time used for a unit of X,

C_Y = cost or time used for a unit of Y, and

a_1 and a_2 = the number of units of X and Y respectively.

In this study N, the total number of subjects is held constant to control for the total df as discussed earlier. Hence, the above cost model can be rewritten as

$$\frac{C_T - NC_0}{N} = a_1 C_X + a_2 C_Y \quad . \tag{5}$$

For more simplicity

$$c = a + b \quad . \tag{6}$$

The $\underline{c}$ in (6) is any available total cost or time of measurement, excluding the fixed cost, C_0. The $\underline{a}$ and $\underline{b}$ are the allocation of cost or time to Y and X respectively. For a simplification and better understanding, as in our psychological studies, c can be conceptualized as the total number of questions in two tests and a and b are the lengths of each test such that $a + b = c$. From now on, the test length concept, where X and Y are psychological tests, is used in place of this cost and time concept. Allocation of test resources using the test length concept, is not different from those of cost or time concepts. This is because, for any test length, time and money have to be spent. For any specified cost or time, it can be transformed to test length or number of items with that cost and time. (cf. Cronbach and Gleser, 1965, p. 329).

By lengthening the test or repeating measurements, higher reliability results with the reduction of the ME variance. The reduction of error variance will affect the power of the F-test as discussed earlier. The concern of this dissertation is the values of a and b that will result in the maximum power of the F-test in ANOCO.

The development starts with placing a restriction on the OSM by considering only two groups, i.e., $i = 1,2$ with equal numbers of subjects in each group. Lord (1960) also used $i = 1,2$ with two repeated measures for the purpose of estimating ME variance. However the restriction in this dissertation is only for a simple development. The discussion of the development without such restriction follows this development. The ME variances, σ_ε^2 and σ_e^2 are assumed either known or estimable, employing methods subsequently described.

The first step in this development is the computation of the F-ratio based on OSM. Following the usual computation procedure (cf. Lindquist, 1953; Myers, 1966; Snedecor and Cochran, 1968), the usual ANOCO table is given in Table 1 with the computation formulas in Table 2.

Table 1. Analysis of Covariance

Sources		df	SS.	F.
Total	Y(adj.)	2n-2	$SS_{Ya} = SS_Y - \dfrac{SP_T^2}{SS_X}$	
Between	(adj.)	1	$SS_{Ba} = SS_{Ya} - SS_{Wa}$	$\dfrac{SS_{Ba}}{SS_{Wa}/(2n-3)}$
Within	(adj.)	2n-3	$SS_{Wa} = SS_{Yw} - \dfrac{SP_w^2}{SS_{Xw}}$	

Thanks to the restriction of $i = 1,2$ repeated measures, the algebraic derivation is simple.

$$SS_Y = \Sigma Y_1^2 + \Sigma Y_2^2 - \frac{(\Sigma Y_1 + \Sigma Y_2)^2}{2n}$$

$$= \Sigma Y_1^2 + \Sigma Y_2^2 - \frac{(\Sigma Y_1)^2 + (\Sigma Y_2)^2 + 2\Sigma Y_1 \Sigma Y_2}{2n}$$

$$= \Sigma Y_1^2 - \frac{(\Sigma Y_1)^2}{n} + \Sigma Y_2^2 - \frac{(\Sigma Y_2)^2}{n} + \frac{2\left[(\Sigma Y_1)^2 + (\Sigma Y_2)^2\right]}{2n}$$

$$- \frac{(\Sigma Y_1)^2 + (\Sigma Y_2)^2 + 2\Sigma Y_1 \Sigma Y_2}{2n}$$

$$= (n-1)S_{Y_1}^2 + (n-1)S_{Y_2}^2 + \frac{(\Sigma Y_1)^2 + (\Sigma Y_2)^2 - 2\Sigma Y_1 \Sigma Y_2}{2n} .$$

Table 2. Computational Formulas for ANOCO

Sources	SS of X	SS of Y	SP of X and Y
Total (T)	$\sum_{i=1}^{2}\sum_{j=1}^{n} X_{ij}^2 - C_X = SS_X$	$\sum_{i=1}^{2}\sum_{j=1}^{n} Y_{ij}^2 - C_Y = SS_Y$	$\sum_{i=1}^{2}\sum_{j=1}^{n} X_{ij}Y_{ij} - C_{XY} = SP_T$
Between (B)	$\sum_{i=1}^{2} (\sum_{j} X_{ij})^2/n - C_X = SS_{Xb}$	$\sum_{i=1}^{2} (\sum_{j} Y_{ij})^2/n - C_Y = SS_{Yb}$	$\sum_{i=1}^{2} (\sum_{j} X_{ij})(\sum_{j} Y_{ij}) - C_{XY} = SP_B$
Within (W)	$SS_X - SS_{Xb} = SS_{Xw}$	$SS_Y - SS_{Yb} = SS_{Yw}$	$SP_T - SP_B = SP_w$
	$C_X = \frac{(\sum\sum X_{ij})}{2n}$	$C_Y = \frac{(\sum\sum Y_{ij})}{2n}$	$C_{XY} = \frac{(\sum\sum X_{ij})(\sum\sum Y_{ij})}{2n}$

By having $S^2_{Y_1} = S^2_{Y_2} = S^2_{Yw}$ and $S^2_{X_1} = S^2_{X_2} = S^2_{Xw}$, i.e., the homogeneity of variance, then

$$SS_Y = 2(n-1)S^2_{Yw} + \frac{(\Sigma Y_1 - \Sigma Y_2)^2}{2n}$$

$$= 2(n-1)S^2_{Yw} + \frac{n(\bar{Y}_1 - \bar{Y}_2)^2}{2} .$$

And similarly,

$$SS_X = 2(n-1)S^2_{Xw} + \frac{n(\bar{X}_1 - \bar{X}_2)^2}{2} .$$

Also

$$SP_T = \Sigma X_1Y_1 + \Sigma X_2Y_2 - \frac{(\Sigma X_1 + \Sigma X_2)(\Sigma Y_1 + \Sigma Y_2)}{2n}$$

$$= \Sigma X_1Y_1 - \frac{\Sigma X_1 \Sigma Y_1}{n} + \Sigma X_2Y_2 - \frac{\Sigma X_2 \Sigma Y_2}{n} + \frac{2(\Sigma X_1 \Sigma Y_1 + \Sigma X_2 \Sigma Y_2)}{2n}$$

$$- \frac{\Sigma X_1 \Sigma Y_1 + \Sigma X_1 \Sigma Y_2 + \Sigma X_2 \Sigma Y_1 + \Sigma X_2 \Sigma Y_2}{2n}$$

$$= (n-1)S^2_{Xw}b_1 + (n-1)S^2_{Xw}b_2 + \frac{(\Sigma X_1 - \Sigma X_2)(\Sigma Y_1 - \Sigma Y_2)}{2n} .$$

By having $b_1 = b_2 = b_w$ i.e., homogeneity of regressions,

$$SP_T = 2(n-1)S^2_{Xw}b_w + \frac{n(\bar{X}_1 - \bar{X}_2)(\bar{Y}_1 - \bar{Y}_2)}{2}$$

$$SP^2_T = \left[2(n-1)S^2_{Xw}b_w + \frac{n(\bar{X}_1 - \bar{X}_2)(\bar{Y}_1 - \bar{Y}_2)}{2}\right]^2 .$$

To find SS_{Wa} we find SS_{Yw}, SP^2_w, and SS_{Xw} as

$$SS_{Wa} = SS_{Yw} - \frac{SP_w^2}{SS_{Xw}} \quad . \quad \text{Hence,}$$

$$SS_{Yw} = \Sigma Y_1^2 - \frac{(\Sigma Y_1)^2}{n} + \Sigma Y_2^2 - \frac{(\Sigma Y_2)^2}{n}$$

$$= 2(n-1)S_{Yw}^2$$

$$SP_w^2 = \left[\Sigma X_1 Y_1 - \Sigma X_1 \Sigma Y_1/n + \Sigma X_2 Y_2 - \Sigma X_2 \Sigma Y_2/n\right]^2$$

$$= \left[2(n-1)S_{Xw}^2 b_w\right]^2 \quad .$$

And $$SS_{Xw} = \Sigma X_1^2 - \frac{(\Sigma X_1)^2}{n} + \Sigma X_2^2 - \frac{(\Sigma X_2)^2}{n}$$

$$= 2(n-1)S_{Xw}^2 \quad .$$

Therefore, $$SS_{Wa} = 2(n-1)S_{Yw}^2 - \frac{\left[2(n-1)S_{Xw}^2 b_w\right]^2}{2(n-1)S_{Xw}^2}$$

$$= 2(n-1)\left[S_{Yw}^2 - S_{Xw}^2 b_w^2\right]$$

$$SS_{Ba} = SS_{Ya} - SS_{Wa}$$

$$= SS_Y - \frac{SP_T^2}{SS_X} - SS_{Wa}$$

$$= 2(n-1)S_{Yw}^2 + \frac{n(\overline{Y}_1-\overline{Y}_2)^2}{2}$$

$$- \frac{\left[2(n-1)S_{Xw}^2 b_w + \frac{n}{2}(\overline{X}_1-\overline{X}_2)(\overline{Y}_1-\overline{Y}_2)\right]^2}{2(n-1)S_{Xw}^2 + \frac{n(\overline{X}_1-\overline{X}_2)^2}{2}} \quad .$$

Algebraically this becomes

$$SS_{Ba} = \frac{n(n-1)S^2_{Xw}\left[b^2_w(\bar{X}_1-\bar{X}_2)^2 + (\bar{Y}_1-\bar{Y}_2)^2 - 2b_w(\bar{X}_1-\bar{X}_2)(\bar{Y}_1-\bar{Y}_2)\right]}{2(n-1)S^2_{Xw} + \frac{n}{2}(\bar{X}_1-\bar{X}_2)^2}$$

$$= \frac{n(n-1)S^2_{Xw}}{2(n-1)S^2_{Xw} + \frac{n}{2}(\bar{X}_1-\bar{X}_2)^2}\left[(\bar{Y}_1-\bar{Y}_2)-b_w(\bar{X}_1-\bar{Y}_2)\right]^2$$

$$= \frac{n(n-1)\left[(\bar{Y}_1-\bar{Y}_2)-b_w(\bar{X}_1-\bar{X}_2)\right]^2}{2(n-1) + n(\bar{X}_1-\bar{X}_2)^2/2S^2_{Xw}} .$$

However $$\frac{n(\bar{X}_1-\bar{X}_2)^2}{2S^2_{Xw}} = t^2_X .$$

Therefore $$SS_{Ba} = \frac{n(n-1)\left[(\bar{Y}_1-\bar{Y}_2) - b_w(\bar{X}_1-\bar{X}_2)\right]^2}{2(n-1) + t^2_X} .$$

Having both SS_{Wa} and SS_{Ba}, the F-ratio for ANOCO can be obtained by the formula in Tables 1 and 2

$$F = \frac{\dfrac{n(n-1)\left[(\bar{Y}_1-\bar{Y}_2) - b_w(\bar{X}_1-\bar{X}_2)\right]^2}{2(n-1) + t^2_X}}{2(n-1)\left[S^2_{Yw}-S^2_{Xw}b^2_w\right]/(2n-3)} .$$

By dividing both numerator and denominator by $(n-1)S^2_{Yw}$,

$$F = \frac{n\left[(\bar{Y}_1-\bar{Y}_2)/S_{Yw} - \dfrac{b_w}{S_{Yw}}(\bar{X}_1-\bar{X}_2)\right]^2/\left[2(n-1) + t^2_X\right]}{2\left[1-\dfrac{S^2_{Xw}}{S^2_{Yw}}b^2_w\right]/[(2n-3)]}$$

$$= \frac{(2n-3)\left[(\bar{Y}_1-\bar{Y}_2)\,\frac{\sqrt{n}}{S_{Yw}} - \frac{b_w}{S_{Yw}}\sqrt{n}\,(\bar{X}_1-\bar{X}_2)\right]^2}{2\left[1-\frac{b_w^2}{S_{Yw}^2}\,S_{Xw}^2\right]\left[2(n-1)+t_X^2\right]}$$

$$= \frac{2(2n-3)\left[(\bar{Y}_1-\bar{Y}_2)\,\frac{\sqrt{n}}{\sqrt{2}\,S_{Yw}} - \frac{b_w S_{Xw}}{S_{Yw}}\sqrt{n}\,\frac{(\bar{X}_1-\bar{X}_2)}{\sqrt{2}\,S_{Xw}}\right]^2}{2\left[1-\frac{b_w^2}{S_{Yw}^2}\,S_{Xw}^2\right]\left[2(n-1)+t_X^2\right]}$$

$$= \frac{2(n-3)\left[t_Y - \frac{S_{Xw}}{S_{Yw}}\,b_w t_X\right]^2}{\left[1-\frac{b_w^2}{S_{Yw}^2}\,S_{Xw}^2\right]\left[2(n-1)+t_X^2\right]}$$

$\frac{S_{Xw}^2}{S_{Yw}^2}$ is a scale factor. With no loss of generality we may assume that an initial scaling has been transformed so that $S_{Xw} = S_{Yw}$. This makes $b=r$. And the F-ratio becomes

$$= \frac{2(n-1)\left[t_Y - rt_X\right]^2}{\left[2(n-1)+t_X^2\right](1-r^2)}$$

$$= \frac{2(n-3)(t_Y^2 - 2rt_Xt_Y + r^2t_X^2)}{\left[2(n-1)+t_X^2\right](1-r^2)} \quad .$$

Since t_X resulted from randomization and _n_ is large, $t_X \sim N(0,1)$, and $E(t_Xt_Y) = E(r_{XY})$ with $E(t_X^2) = 1$.

Therefore, asymptotically

$$F = (t_Y^2 - r^2)/(1-r^2) = (F_Y - r^2)/(1-r^2) \quad .$$

This F-value has been derived with the restriction of two treatment groups and one covariate, X. However, with more than two groups, the derivation results in the same formula. For example with three groups, the sum of squares can be found as follows:

$$SS_Y = \Sigma Y_1 + \Sigma Y_2 + \Sigma Y_3 - \frac{(\Sigma Y_1 + \Sigma Y_2 + \Sigma Y_3)^2}{3n}$$

$$= \Sigma Y_1^2 + \Sigma Y_2^2 + \Sigma Y_3^2$$

$$- \left[\frac{(\Sigma Y_1)^2 + (\Sigma Y_2)^2 + (\Sigma Y_3)^2 + 2\Sigma Y_1 \Sigma Y_2 + 2\Sigma Y_2 \Sigma Y_3 + 2\Sigma Y_3 \Sigma Y_1}{3n} = A \right]$$

$$= \Sigma Y_1^2 - \frac{(\Sigma Y_1)^2}{n} + \Sigma Y_2^2 - \frac{(\Sigma Y_2)^2}{n} + \Sigma Y_3^2 - \frac{(\Sigma Y_3)^2}{n}$$

$$+ \frac{3}{3} \; \frac{(\Sigma Y_1)^2 + (\Sigma Y_2)^2 + (\Sigma Y_3)^2}{n} - [A]$$

$$= 3(n-1)S_{Yw}^2 + \frac{(\Sigma Y_1 - \Sigma Y_2)^2 + (\Sigma Y_2 - \Sigma Y_3)^2 + (\Sigma Y_1 - \Sigma Y_3)^2}{3n}$$

$$= 3(n-1)S_{Yw}^2 + n \left[\frac{(\bar{Y}_1 - \bar{Y}_2) + (\bar{Y}_2 - \bar{Y}_3) + (\bar{Y}_1 - \bar{Y}_3)}{3} \right] \quad .$$

The above SS_Y for three treatment groups is equivalent to that of the two groups previously, but has the factor of 3 instead of 2. Also, there are ${}^3C_2$ comparisons of the means instead of ${}^2C_2$ for two treatment groups. The other sum of squares and sum of products follow the same pattern as discussed above. For $i = i,2,3$ the final forms are:

$$SS_{Ba} = \frac{n(n-1)}{3(n-1) + \frac{2}{3}F_X} \Big[(\bar{Y}_1-\bar{Y}_2)+(\bar{Y}_2-\bar{Y}_3)+(\bar{Y}_1-\bar{Y}_3)$$

$$- b_w\left\{ (\bar{X}_1-\bar{X}_2)+(\bar{X}_2-\bar{X}_3)+(\bar{X}_1-\bar{X}_3)\right\}\Big]^2$$

$$SS_{Wa} = 3(n-1)\left[S^2_{Yw}-S^2_{Xw}b^2_w\right] .$$

For any k groups, the multiple becomes k instead of 3 as above. The b_w or r is now the average of all within group regression coefficients. The F_X is also the average of all possible t^2_X in consideration. Therefore, asymptotically

$$F = \frac{F_Y-r^2}{1-r^2} ,$$

which is the same as the formula derived earlier from just two groups.

$$\text{But} \quad E(t^2_Y) = \frac{\sigma^2_Y + \sigma^2_e + nK^2_\alpha}{\sigma^2_y + \sigma^2_e}$$

$$\text{and} \quad E(r^2) = \frac{\sigma^2_{XY}}{(\sigma^2_y + \sigma^2_e)(\sigma^2_x + \sigma^2_\varepsilon)} .$$

Hence, asymptotically

$$F = \frac{F_Y-r^2}{1-r^2}$$

$$= \frac{\sigma_y^2+\sigma_e^2+nK_\alpha^2)/(\sigma_y^2+\sigma_e^2)-\sigma_{XY}^2/(\sigma_y^2+\sigma_e^2)(\sigma_x^2+\sigma_\varepsilon^2)}{1-\sigma_{XY}^2/(\sigma_y^2+\sigma_e^2)(\sigma_x^2+\sigma_\varepsilon^2)}$$

$$= 1 + \frac{nK_\alpha^2/(\sigma_y^2+\sigma_e^2)-\sigma_{XY}^2/(\sigma_y^2+\sigma_e^2)(\sigma_x^2+\sigma_\varepsilon^2)}{1-\sigma_{XY}^2/(\sigma_y^2+\sigma_e^2)(\sigma_x^2+\sigma_\varepsilon^2)}$$

$$= 1 + \frac{nK_\alpha^2(\sigma_x^2+\sigma_\varepsilon^2)}{(\sigma_y^2+\sigma_e^2)(\sigma_x^2+\sigma_\varepsilon^2)-\sigma_{XY}^2}$$

$$= 1 + \frac{nk_\alpha^2}{\sigma_Y^2+\sigma_e^2-\sigma_{XY}^2/(\sigma_x^2+\sigma_\varepsilon^2)} = 1 + \frac{nk_\alpha^2}{K} = 1 + nk_\alpha^2 T$$

for $K = \sigma_Y^2+\sigma_e^2-\sigma_{XY}^2/(\sigma_x^2+\sigma_\varepsilon^2) = \frac{1}{T}$. This is for a unit test length of Y and X.

For a specified available resource, say the number of total test items is $\underline{c}$, tests Y and X are lengthened by the factor $\underline{a}$ and $\underline{b}$ respectively such that the total items on the two tests equals $\underline{c}$. In that case

$$K = \sigma_y^2 + \frac{\sigma_e^2}{a} - \frac{\sigma_{XY}^2}{\sigma_x^2+\sigma_\varepsilon^2/(c-a)} .$$

To find $\underline{a}$ that will maximize the power of the F-test is equivalent to finding $\underline{a}$ in K such that K is minimized.

By differentiating K with respect to $\underline{a}$ and letting the result equal to zero, we can solve for $\underline{a}$. That is

$$\frac{dk}{da} = -\frac{\sigma_e^2}{a^2} + \frac{\sigma_{XY}^2\ \sigma_\varepsilon^2}{\left[(c-a)\,\sigma_x^2 + \sigma_\varepsilon^2\right]^2} = 0$$

$$a^2\left[\sigma_{XY}^2\sigma_\varepsilon^2-\sigma_x^4\sigma_e^2\right] + a\left[2\sigma_x^2\sigma_e^2(c\sigma_x^2+\sigma_\varepsilon^2)\right] - \sigma_e^2(c\sigma_x^2+\sigma_\varepsilon^2)^2 = 0.$$

By specifying the variance of X and Y as $\sigma_x^2+\sigma_\varepsilon^2$ and $\sigma_y^2+\sigma_e^2$ as both equal to 1, algebraically an optimum allocation denoted by a_0 instead of $\underline{a}$ results:

$$a_0 = \frac{c+\sigma_\varepsilon^2/(1-\sigma_\varepsilon^2)}{1 \pm\sigma_\varepsilon\sigma_{XY}/\sigma_e(1-\sigma_\varepsilon^2)} .$$

The $\pm$ sign takes the same value as the value of σ_{XY} so that the denominator is never zero. The above σ_{XY} is the observed correlation. The true correlation can be found with the following relationship

$$\sigma_{xy} = \frac{\sigma_{XY}}{(1-\sigma_e^2)^{1/2}(1-\sigma_\varepsilon^2)^{1/2}} .$$

Then

$$\sigma_{XY} = \sigma_{xy}(1-\sigma_e^2)^{1/2}(1-\sigma_\varepsilon^2)^{1/2} .$$

Using the above information the equation for $\underline{a_0}$ can be expressed as

$$a_0 = \frac{c+\sigma_\varepsilon^2/(1-\sigma_\varepsilon^2)}{1\pm\sigma_{xy}\sigma_\varepsilon/(1-\sigma_\varepsilon^2)^{1/2} \cdot (1-\sigma_e^2)^{1/2}/\sigma_e} .$$

In terms of their reliabilities, the above becomes

$$a_0 = \frac{c + \dfrac{1-r_{XX}}{r_{XX}}}{1+\sigma_{xy}\left[\dfrac{r_{YY}(1-r_{XX})}{r_{XX}(1-r_{YY})}\right]^{1/2}}$$

For the case of more than one covariate X, the derivation seems to be not so simple. However, the number of covariates is not the question of this dissertation. This is because our intention is to allocate available resources to both variates and covariates in such a way that maximum power is obtained. Given a composite variate and/or covariate, the input into these results must be σ_e^2, σ_ε^2, σ_{xy} for those composites. The problem of how to allocate resources to each element in the composite goes beyond the scope and purpose of this dissertation. For further detailof this problem, the works by Horst (1956) as well as Woodbury and Novick (1968) are recommended.

Discussion of the Formula of Allocation

From the above formula, $\underline{a_0}$ is a function of three variables; the reliabilities of X and Y, r_{XX} and r_{YY} (or σ_ε^2 and σ_e^2), and their correlation, σ_{xy} . Considering this formula, some interesting points can be seen. If $r_{XX} \to 1$ or $\sigma_\varepsilon^2 \to 0$ most of the resource is allocated to Y. The same analogy applies with $r_{YY} \to 1$. This implies that with a perfect measurement on either variable, no repeated measure is needed on that variable. Therefore, the resource should be allocated to the other variable. In other words, repeated measurement is needed for a variable such that its reliability will be enhanced. Whenever the correlation

$\sigma_{xy} \to 0$, the formula tells us that most of the resource should be allocated to Y. This conclusion is obvious as discussed by Huck (1972), that ANOCO is superior to ANOVA by reducing the variability in Y. But if $\sigma_{xy} = 0$, such superiority is not obtained. That is, the variability of $\hat{Y} = \alpha + \beta X$ is the same as that of Y whenever $\beta = 0$.

To see if the derived formula does make further sense, we first compute the values of $\underline{a_0}$ at some selected combinations of σ_ε^2, σ_e^2, and σ_{xy}^2 when each one takes the specified values .1, .2,...,.9 with the overall total repeated measure of 100. The values of $\underline{a_0}$ are displayed in Table 3.

In this equation the value of $c = 100$ was chosen after considering the values selected for σ_e^2, σ_ε^2, and σ_{xy}^2. With $c = 1000$, for example, even $\sigma_e^2 = \sigma_\varepsilon^2 = .9$ would not be large since about 500 repeated measures of X and Y would result in little error. Similarly, if $c = 10$, then the error involved even when $\sigma_e^2 = \sigma_\varepsilon^2 = .1$ would be substantial.

Table 3 would seem most useful for the researcher. He could determine σ_e^2 and σ_ε^2 which would result from .01 of his total resources allotted to each. For example, if he knew that a 60 item intelligence test had a reliability of .9 and a 30 item achievement test had a reliability of .8 and that the two tests correlated .6, and if both tests took about one hour to administer and he could afford 4 hours for testing, he could consider a 2.4 minute interval for estimating

σ_e^2 and σ_ε^2 .

For the intelligence test

$$r_{XX} = .9 = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\varepsilon^2 / \frac{60}{2.4}} = \frac{1-\sigma_\varepsilon^2}{1-\sigma_\varepsilon^2+\sigma_\varepsilon^2/25} \quad , \quad \text{thus}$$

$$\sigma_\varepsilon^2 \simeq .70 \quad .$$

Similarly

$$r_{YY} = .8 = \frac{\sigma_y^2}{\sigma_y^2 + \sigma_e^2 / \frac{30}{2.4}}$$

$$\sigma_e^2 = .75 \quad .$$

Also, $\sigma_{xy}^2 = \frac{.36}{.9 \cdot .8} = .5$.

Using these values in Table 3, one finds for

$\sigma_\varepsilon^2 = .7$, $\sigma_e^2 = .7$, $\sigma_{xy}^2 = .5$ that $a_0 = 60$ and for

$\sigma_\varepsilon^2 = .7$, $\sigma_e^2 = .8$, $\sigma_{xy}^2 = .5$ that $a_0 = 66$. Thus one should test achievement for 63 x 2.4 minutes and test intelligence for 37 x 2.4 minutes. In terms of items, the achievement test should contain 76 items and the intelligence test should contain 89 items.

Table 3. Values of a_0 , the optimum allocation to Y for some combinations of σ^2_ε σ^2_e and σ^2_{xy} where c = 100

		σ^2_{xy}								
σ^2_ε	σ^2_e	.1	.2	.3	.4	.5	.6	.7	.8	.9
.1	.1	76	69	65	61	59	56	55	53	51
.1	.2	83	77	73	70	68	66	64	63	61
.1	.3	86	82	78	76	74	72	70	69	68
.1	.4	89	85	82	80	79	76	75	73	72
.1	.5	91	87	85	83	81	80	78	77	76
.1	.6	92	89	87	85	84	83	82	81	80
.1	.7	94	91	89	88	87	86	85	84	83
.1	.8	95	93	92	91	90	89	88	87	86
.1	.9	97	95	94	94	93	92	92	91	91
.2	.1	69	60	55	51	49	46	44	43	41
.2	.2	76	69	65	61	59	56	55	53	51
.2	.3	81	75	71	68	65	63	61	60	58
.2	.4	84	79	75	72	70	68	66	65	63
.2	.5	87	82	79	76	74	72	71	69	68
.2	.6	89	85	82	80	78	76	75	73	72
.2	.7	91	87	85	83	81	80	79	78	76
.2	.8	93	90	88	87	85	84	83	82	81
.2	.9	95	93	91	91	90	89	88	87	87

Table 3. (Continued)

		σ^2_{xy}								
σ^2_ε	σ^2_e	.1	.2	.3	.4	.5	.6	.7	.8	.9
.3	.1	62	53	48	45	42	40	38	36	35
.3	.2	71	63	58	55	52	50	48	46	45
.3	.3	76	69	65	62	59	57	55	53	52
.3	.4	80	74	70	67	64	62	60	58	57
.3	.5	83	78	74	71	69	67	65	63	62
.3	.6	86	81	78	75	73	71	69	68	67
.3	.7	88	84	81	79	77	75	74	73	71
.3	.8	91	88	85	83	82	80	79	78	77
.3	.9	94	91	90	88	87	86	85	84	83
.4	.1	57	48	43	39	37	35	33	32	30
.4	.2	66	58	53	50	47	44	43	41	39
.4	.3	72	65	60	56	53	51	49	48	46
.4	.4	76	70	65	62	59	57	55	53	52
.4	.5	80	74	70	66	64	62	60	58	57
.4	.6	83	78	74	71	68	66	65	63	62
.4	.7	86	81	78	75	73	71	70	68	67
.4	.8	89	85	82	80	78	76	75	74	73
.4	.9	93	90	88	86	84	83	82	81	80

Table 3. (Continued)

		σ^2_{xy}								
σ^2_ε	σ^2_e	.1	.2	.3	.4	.5	.6	.7	.8	.9
.5	.1	52	43	38	35	32	30	29	27	26
.5	.2	62	53	48	45	42	40	38	36	35
.5	.3	68	60	55	51	49	46	44	43	41
.5	.4	73	65	60	57	54	52	50	48	47
.5	.5	77	70	65	62	59	57	55	53	52
.5	.6	80	74	70	67	64	62	60	58	57
.5	.7	84	78	74	71	69	67	65	64	62
.5	.8	87	83	79	77	75	73	71	70	69
.5	.9	91	88	85	83	82	80	79	78	77
.6	.1	47	38	34	31	28	26	25	24	23
.6	.2	57	48	43	40	37	35	33	32	31
.6	.3	64	55	50	46	44	41	40	38	37
.6	.4	69	61	56	52	49	47	45	43	42
.6	.5	73	66	61	57	54	52	50	48	47
.6	.6	77	70	66	62	59	57	55	54	52
.6	.7	81	75	71	67	65	63	61	59	58
.6	.8	85	80	76	73	71	69	67	66	64
.6	.9	90	86	83	81	79	77	76	74	73

Table 3. (Continued)

		σ^2_{xy}								
σ^2_{ε}	σ^2_e	.1	.2	.3	.4	.5	.6	.7	.8	.9
.7	.1	42	34	29	26	24	22	21	20	19
.7	.2	52	43	38	35	32	30	29	27	26
.7	.3	59	50	45	41	39	36	35	33	32
.7	.4	64	56	51	47	44	42	40	38	37
.7	.5	69	61	56	52	49	47	45	43	42
.7	.6	73	66	61	57	54	52	50	48	47
.7	.7	78	71	66	63	60	58	56	54	53
.7	.8	82	76	72	69	66	64	62	61	59
.7	.9	88	83	80	77	75	73	72	70	69
.8	.1	36	28	24	22	20	18	17	16	16
.8	.2	46	37	33	29	27	25	24	23	22
.8	.3	53	44	39	35	33	31	29	28	27
.8	.4	59	50	44	41	38	26	34	33	31
.8	.5	64	55	50	46	43	41	39	37	36
.8	.6	69	60	55	51	48	46	44	42	41
.8	.7	74	66	61	57	54	51	50	48	46
.8	.8	79	72	67	64	61	59	57	55	53
.8	.9	86	80	76	73	71	69	67	65	64

Table 3. (Continued)

		σ^2_{xy}								
σ^2_ε	σ^2_e	.1	.2	.3	.4	.5	.6	.7	.8	.9
.9	.1	28	22	18	16	15	14	13	12	11
.9	.2	38	30	25	23	21	19	18	17	16
.9	.3	45	36	31	28	26	24	23	21	20
.9	.4	50	41	36	33	30	28	27	25	24
.9	.5	56	47	41	38	35	33	31	30	28
.9	.6	61	52	47	43	40	38	36	34	33
.9	.7	67	58	53	49	46	43	41	40	38
.9	.8	74	65	60	56	53	50	48	47	45
.9	.9	83	75	70	67	64	61	59	58	56

Graphic representations of the relationship of $\underline{a_0}$ with σ^2_e, σ^2_ε and σ^2_{xy} variables are shown in Figures 1-10. Figures 1-9 demonstrate the same relationship with different emphasis. Figure 1 exhibits the values of $\underline{a_0}$ as the function of σ^2_e at three values of σ^2_ε, .1, .5 and .9 at $\sigma^2_{xy} = .1$. It can be seen that at each σ^2_ε, the value of $\underline{a_0}$ increases with increasing values of σ^2_e. Comparing the value of $\underline{a_0}$ at different values of σ^2_ε, the graphs in Figure 1 indicate that higher values of $\underline{a_0}$ occur with lower values of σ^2_ε. This is

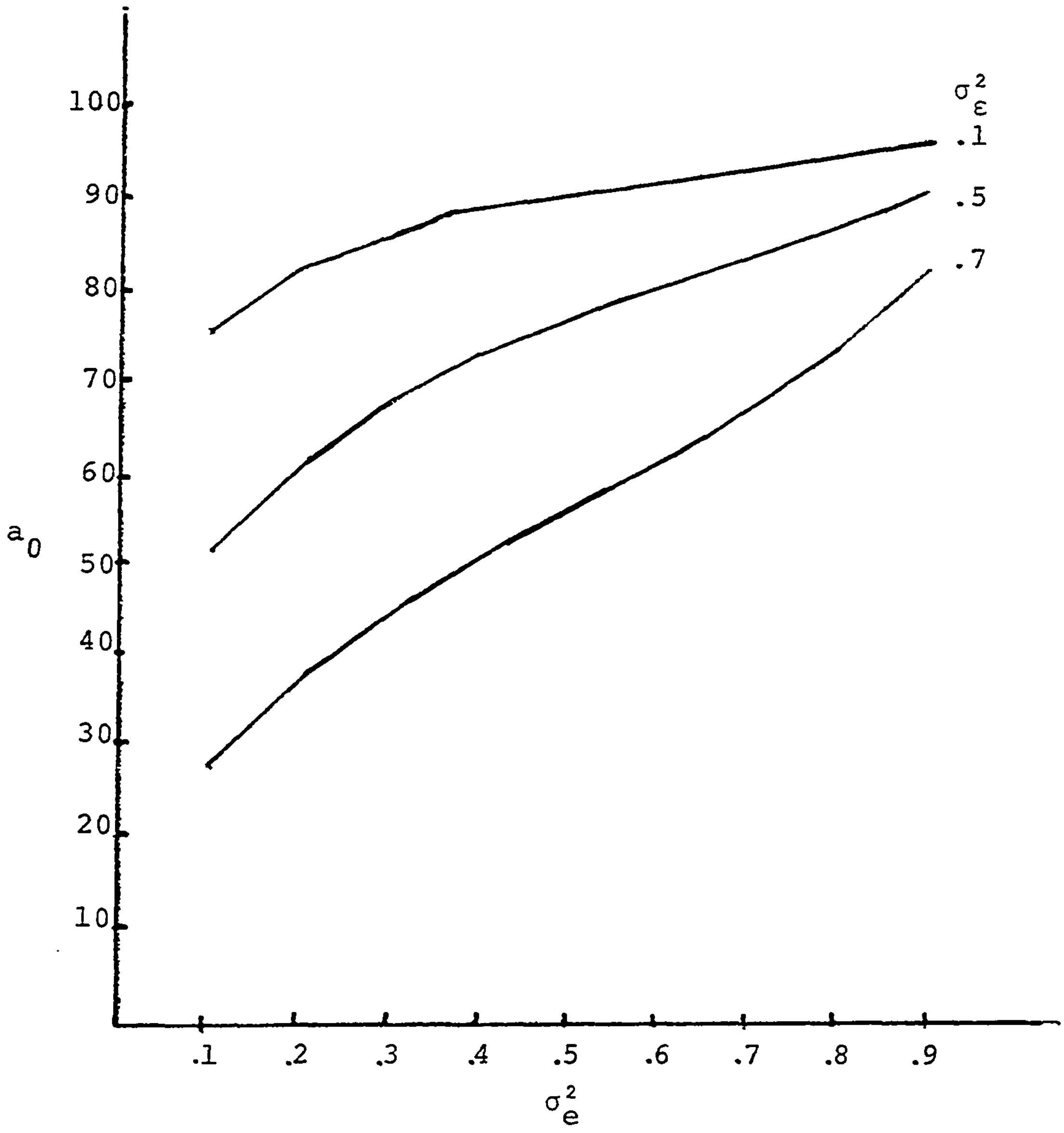

Figure 1. Relation of optimum allocation for Y, a_0, to the ME variance of Y, σ_e^2 for $\sigma_\varepsilon^2 = .1, .5$, and .9 at $\sigma_{xy}^2 = .1$

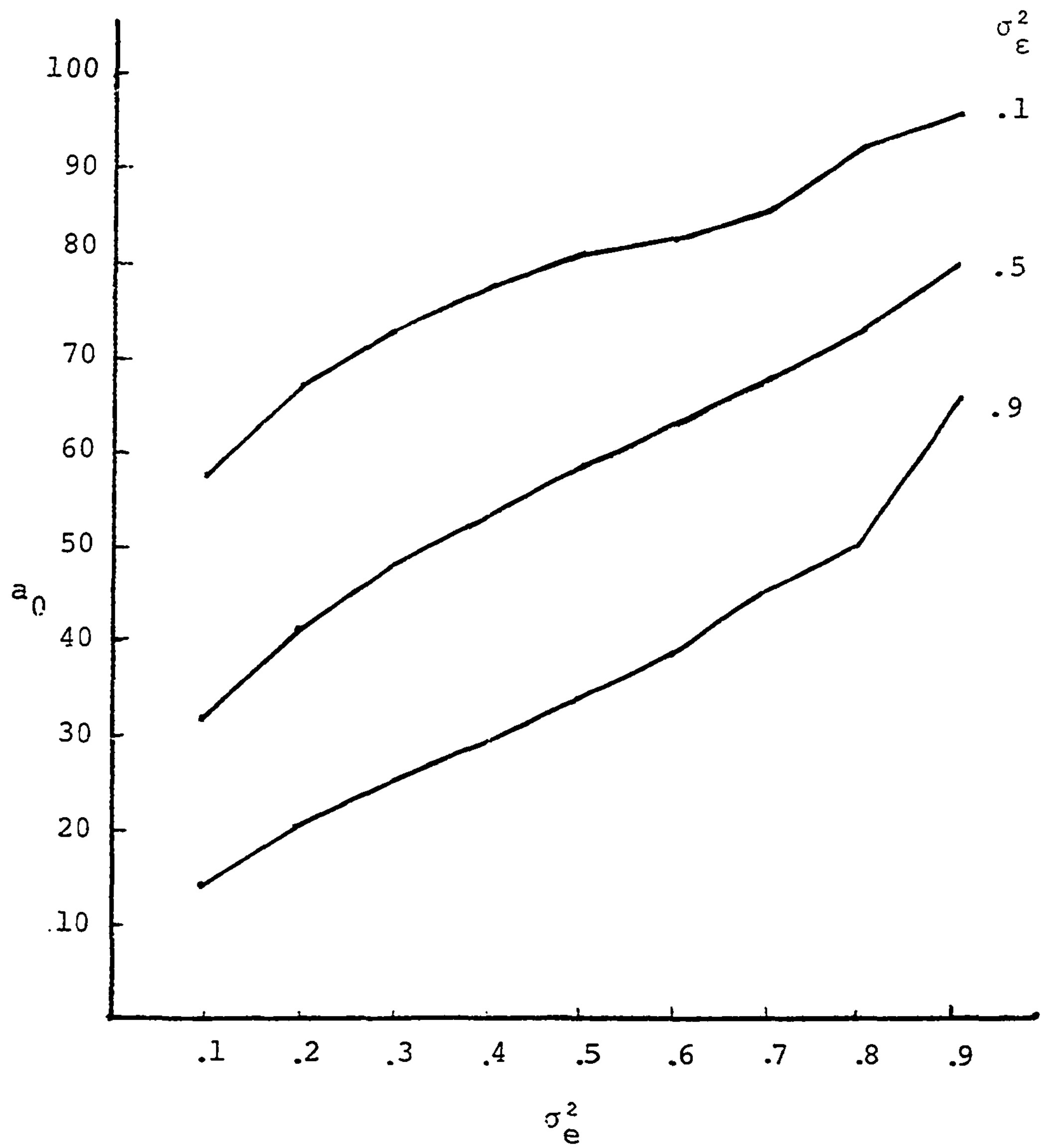

Figure 2. Relation of optimum allocation for Y, a_0 to the ME variance of Y, σ_e^2 for $\sigma_\varepsilon^2 = .1, .5,$ and $.9$ at $\sigma_{xy}^2 = .5$

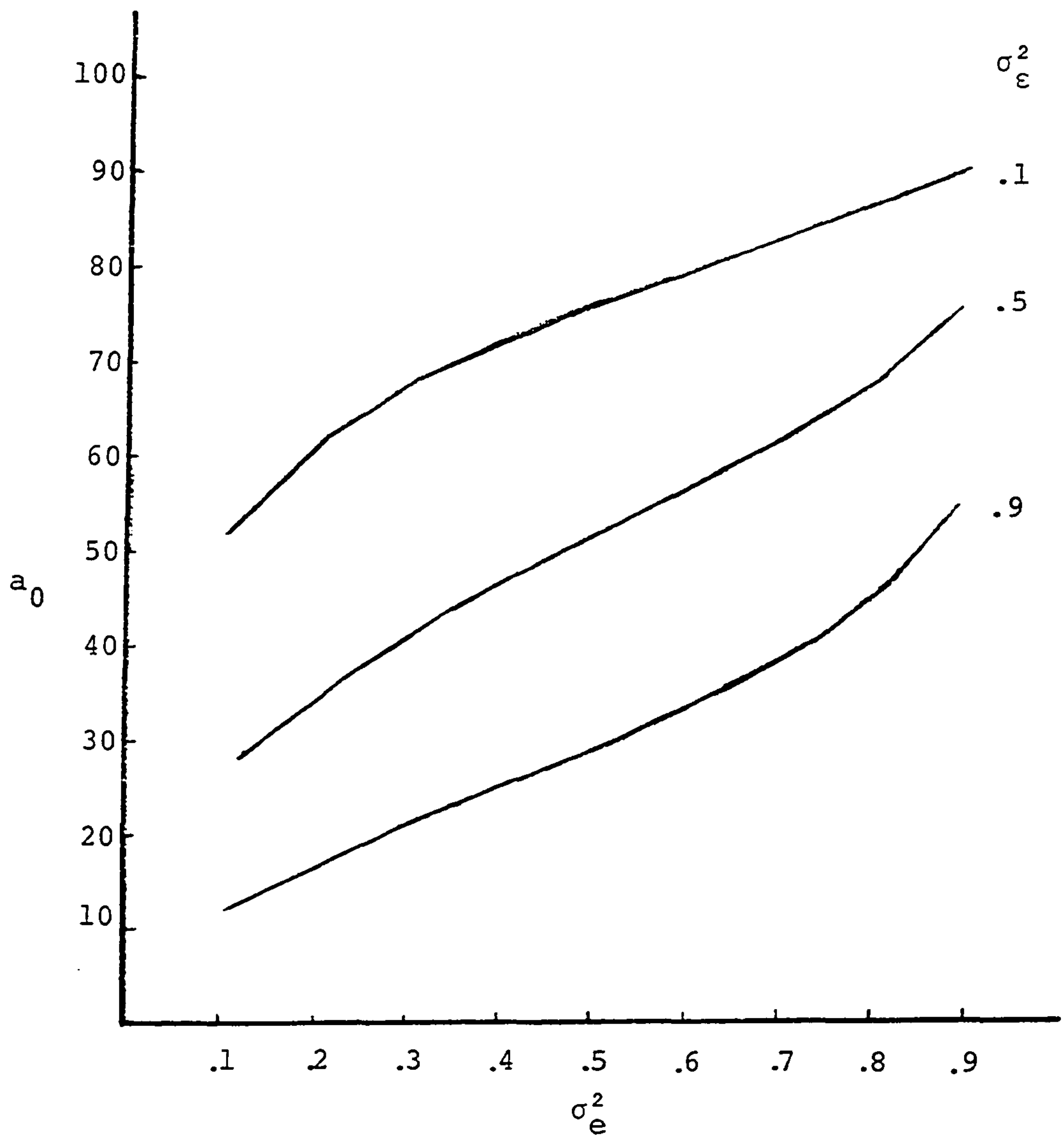

Figure 3. Relation of optimum allocation for Y, a_0 to the ME variance of Y, σ_e^2 for $\sigma_\varepsilon^2 = .1, .5$, and $.9$ at $\sigma_{xy}^2 = .9$

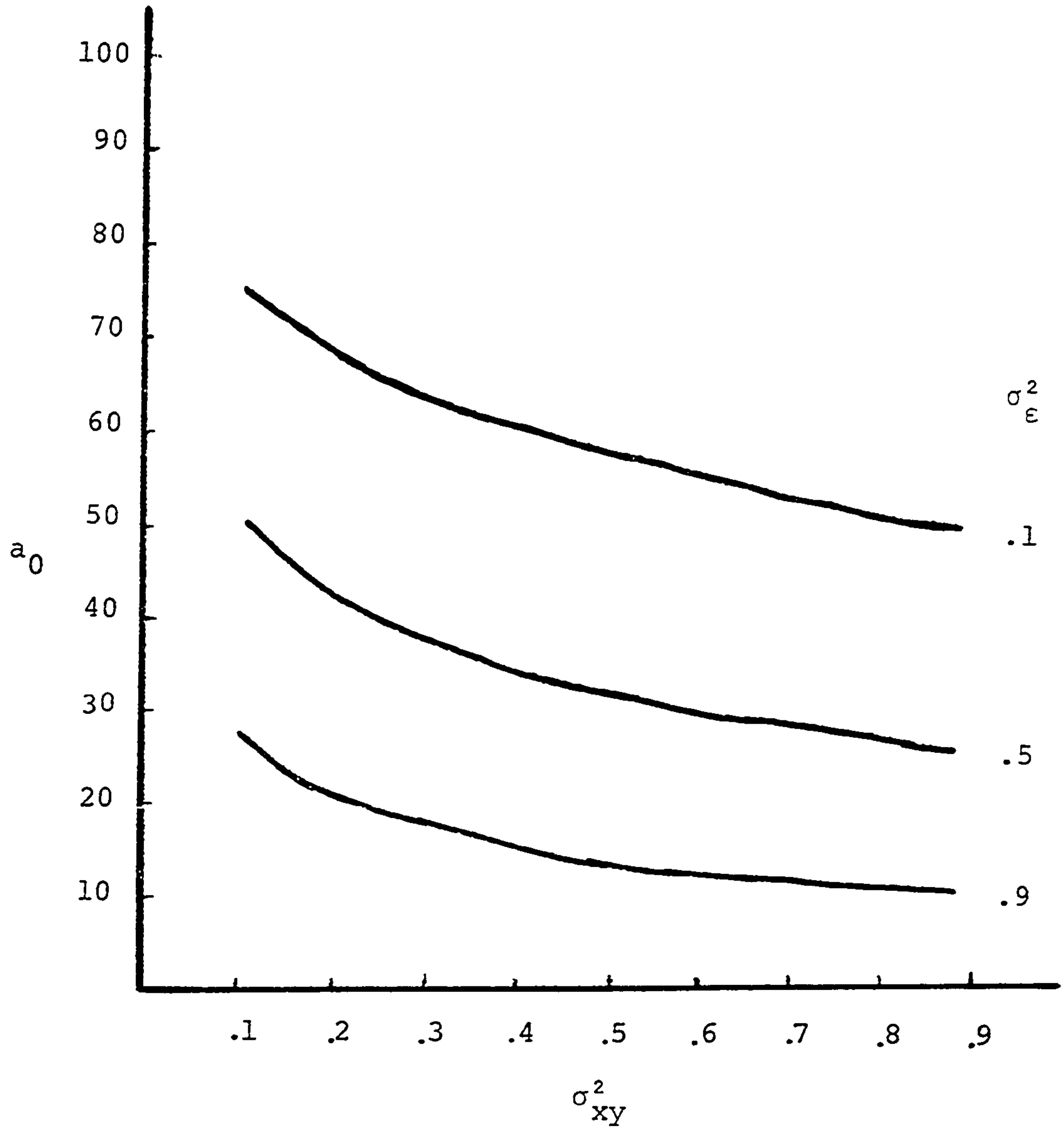

Figure 4. Relation of optimum allocation for Y, a_0 to the squared correlation, σ^2_{xy} for $\sigma^2_\varepsilon = .1$, .5, and .9 at $\sigma^2_e = .1$

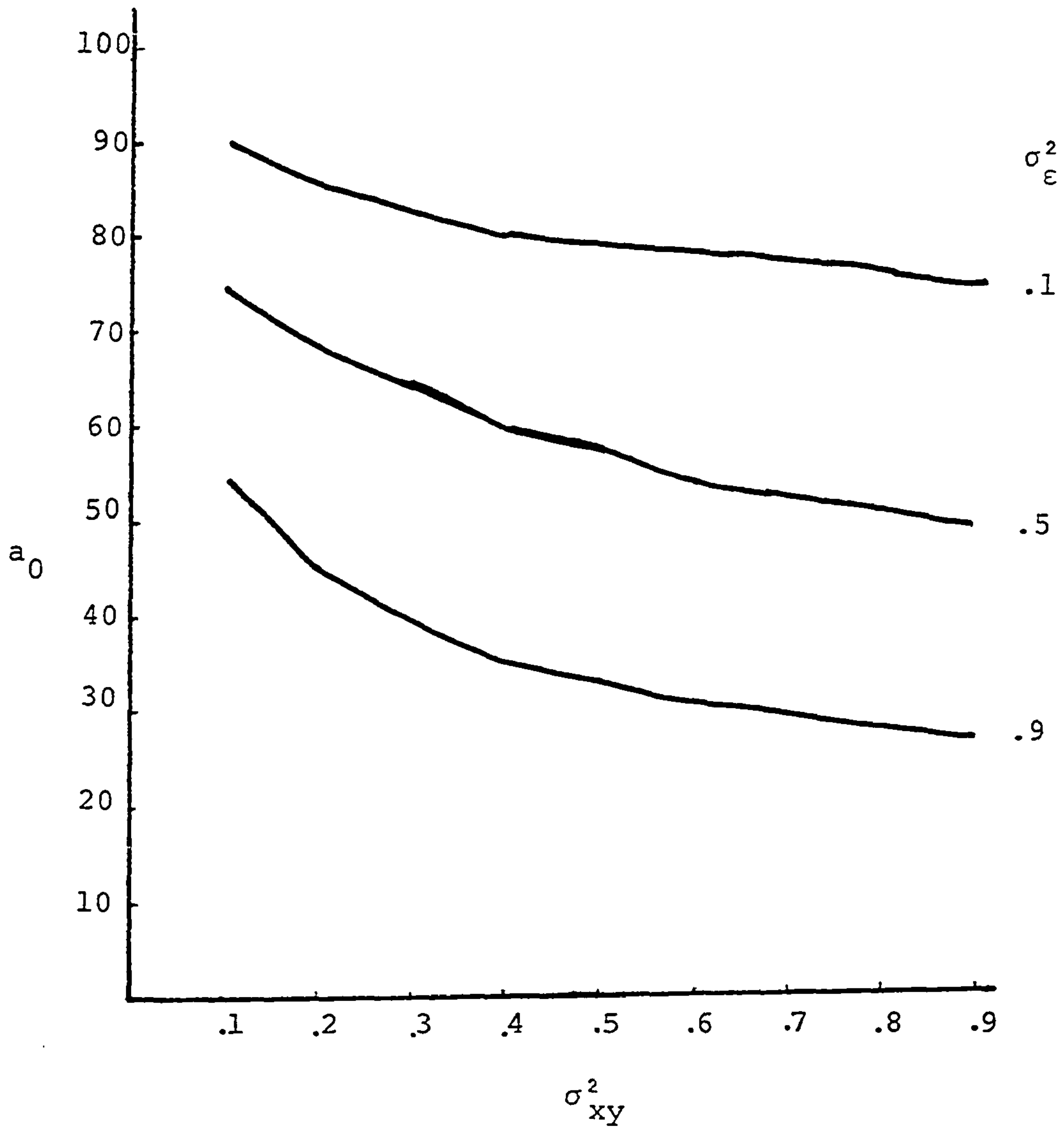

Figure 5. Relation of optimum allocation for Y, a_0 to the squared correlation, σ^2_{xy} for σ^2_ε = .1, .5, and .9 at σ^2_e = .5

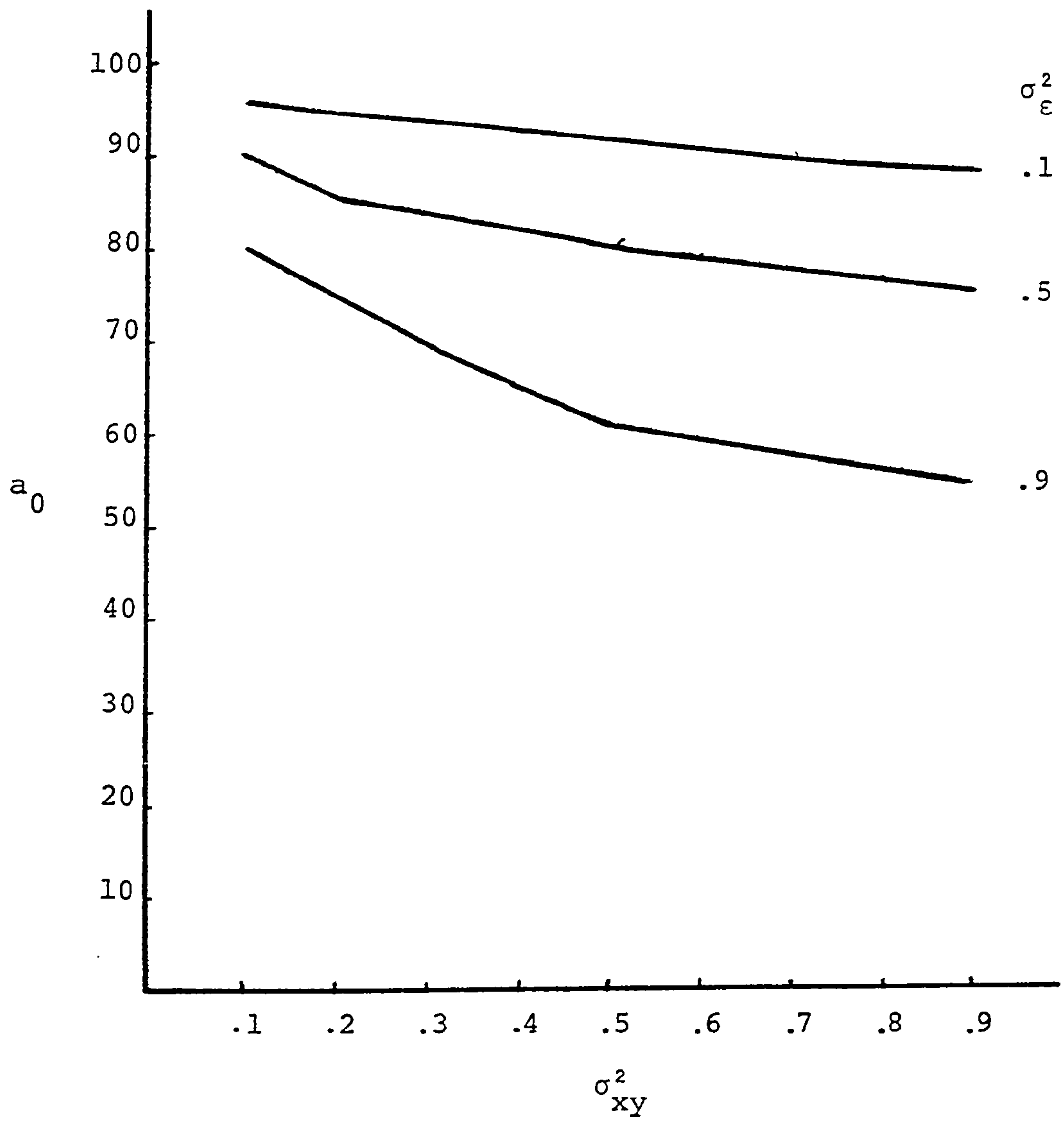

Figure 6. Relation of optimum allocation for Y, a_0 to the squared correlation, σ^2_{xy} for σ^2_ε = .1, .5, and .9 at σ^2_e = .9

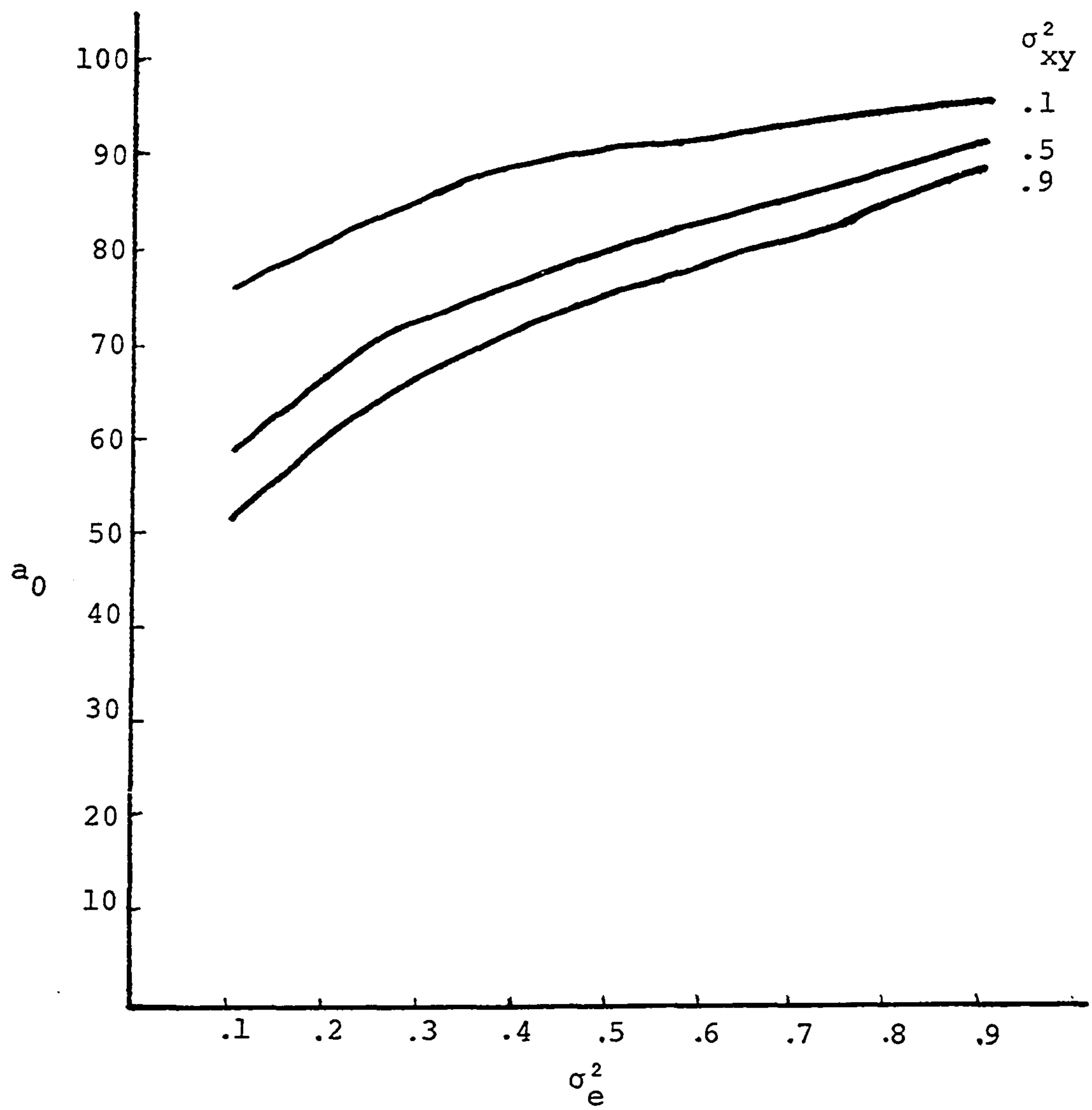

Figure 7. Relation of optimum allocation for Y, a_0 to the ME variance of Y, σ^2_e for $\sigma^2_{xy} = .1$, .5, and .9 at $\sigma^2_\varepsilon = .1$

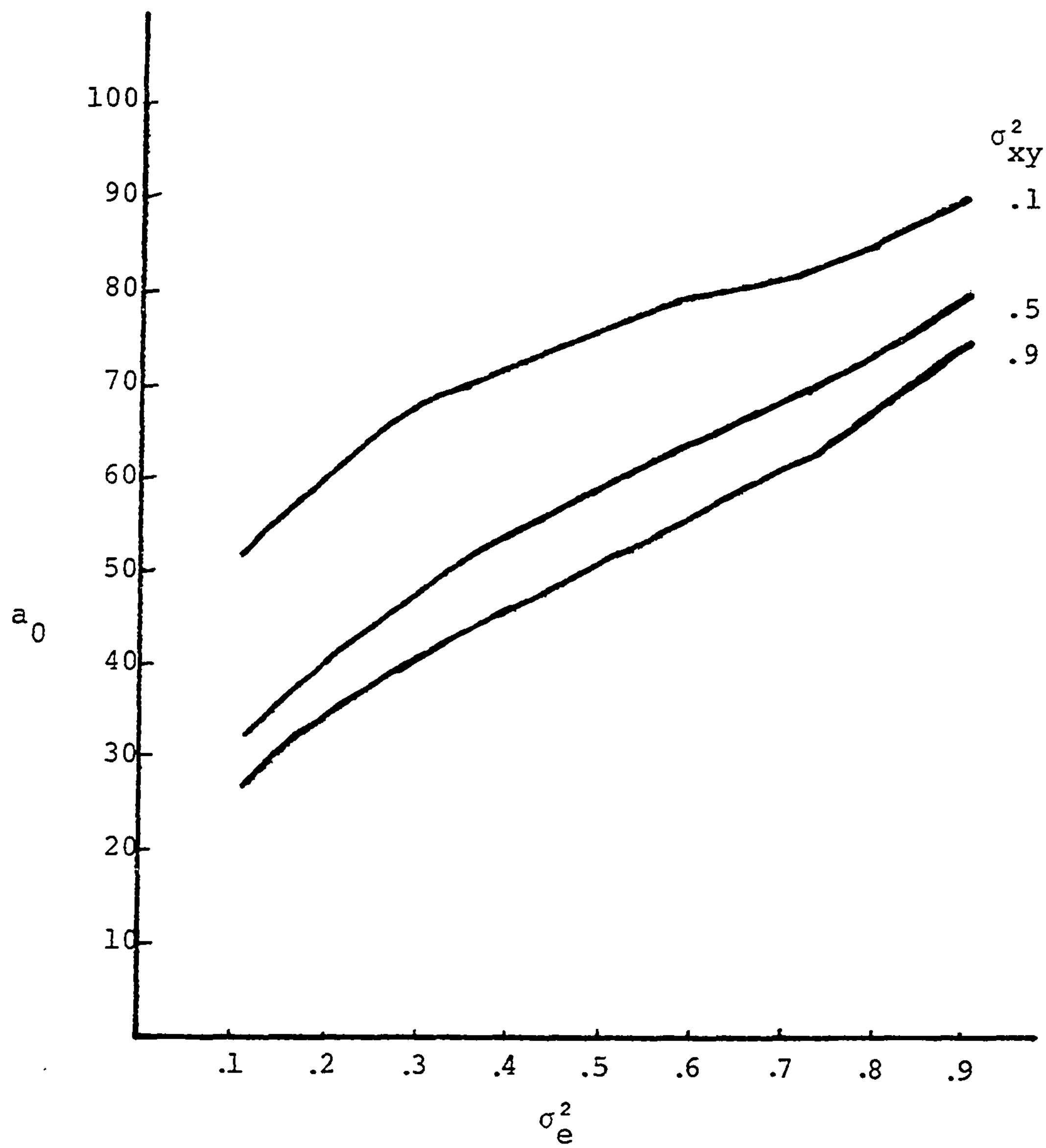

Figure 8. Relation of optimum allocation for Y, a_0 to the ME variance of Y, σ^2_e for σ^2_{xy} = .1, .5, and .9 at σ^2_ε = .5

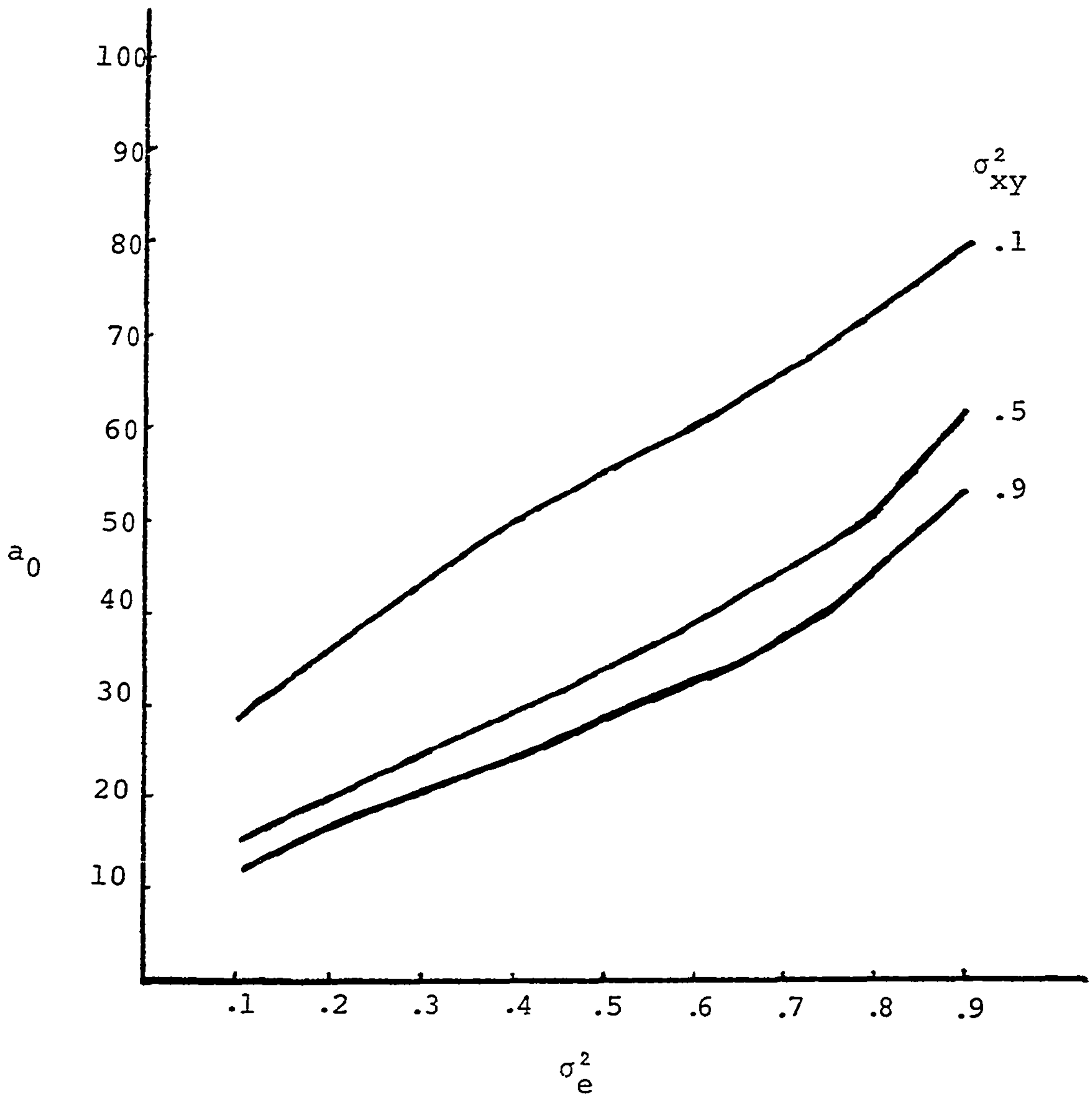

Figure 9. Relation of optimum allocation for Y, a_0 to the ME variance of Y, σ^2_e for σ^2_{xy} = .1, .5, and .9 at σ^2_ε = .9

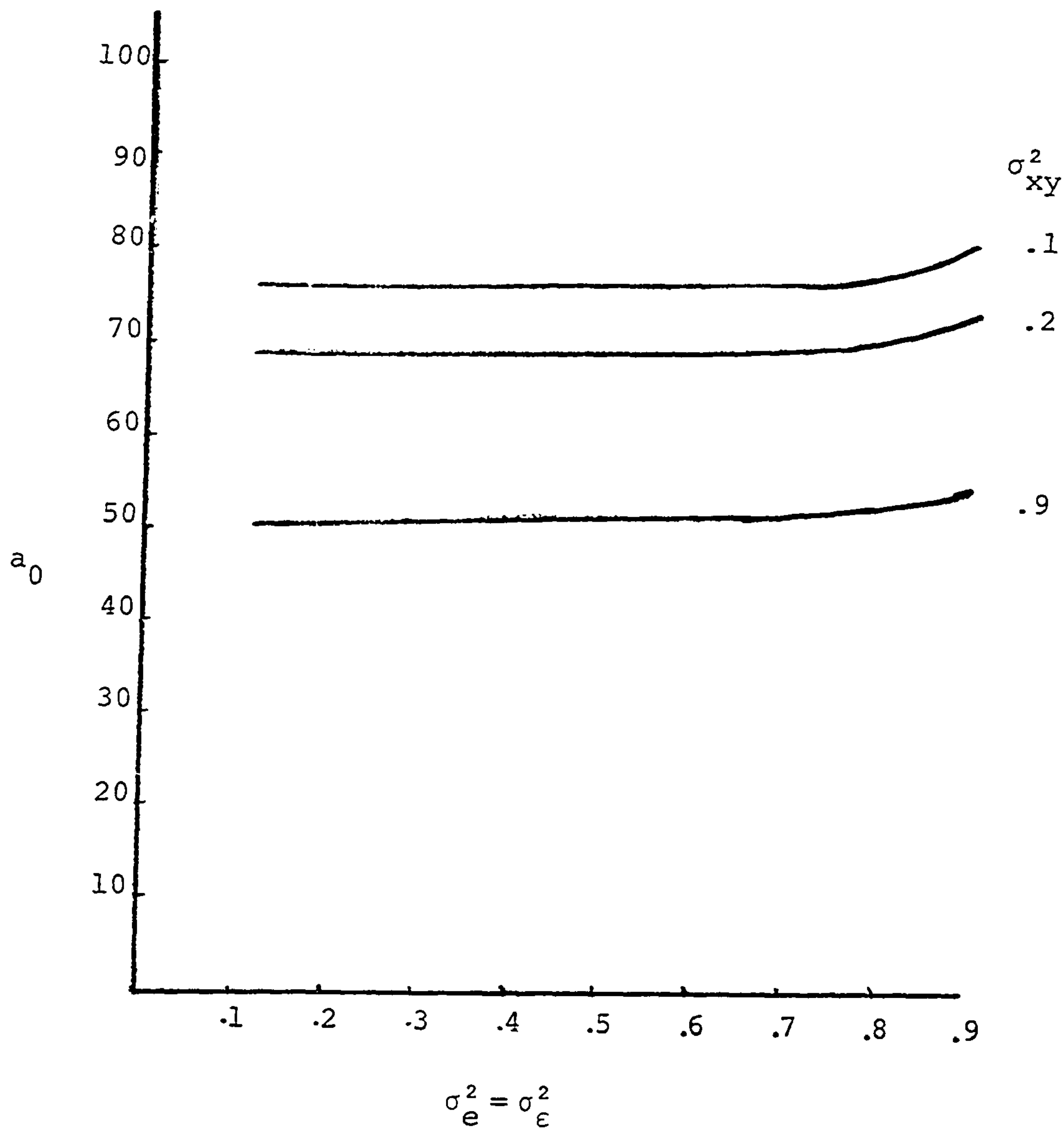

Figure 10. Relation of optimum allocation for Y, a_0 to σ^2_e and σ^2_ε when $\sigma^2_e = \sigma^2_\varepsilon$ for σ^2_{xy} = .1, .2, and .9

accordance with the previous discussion that as $\sigma_\varepsilon^2 \to 0$ most allocation is for Y.

Figures 2 and 3 depict results similar to Figure 1 but at two different values of σ_{xy}^2, .5, and .9, respectively. These graphs in both Figures 2 and 3 have the same interpretation as of Figure 1. The comparison of graphs of these three Figures 1, 2, and 3 depicts how the relationship of $\underline{a_0}$ to σ_e^2 varies with varied values of σ_{xy}^2. In general it appears for all values of σ_e^2 and σ_ε^2, $\underline{a_0}$ is larger when σ_{xy}^2 is smaller. This effect of variation in σ_{xy}^2 is also indicated by graphs in Figures 4-6. These figures show the values of $\underline{a_0}$ as a function of σ_{xy}^2 with nine combinations of σ_e^2 and σ_ε^2 when each one takes the values .1, .5, and .9 . The effect of σ_{xy}^2 in general, is that with lower σ_{xy}^2 the more allocation is for Y. This is shown in all three figures, i.e., each graph in the figures shows decreasing $\underline{a_0}$ with higher σ_{xy}^2. The comparisons for different values of σ_e^2 are the same as discussed for Figures 1-3.

Figures 7-9 contain nothing more than the information from the above six figures. The interpretation is that with higher value of σ_ε^2, the lower is the allocation for Y and the greater the allocation for X. More interesting are the graphs in Figure 10. This figure displays the values of $\underline{a_0}$ at three values of σ_{xy}^2 when $\sigma_e^2 = \sigma_\varepsilon^2$. At different values of σ^2_{xy} , these graphs indicate that with higher σ^2_{xy} , the lower is the allocation for Y. But in no case is the optimum allocation for Y lower than X when $\sigma^2_e = \sigma^2_\varepsilon$. For example, with $\sigma^2_{xy} \to 1$, the two allocations are nearly half and half, but with the allocation for Y approaching $\underline{c}$ as both σ^2_e and $\sigma^2_\varepsilon \to 1$.

In conclusion of this section of the dissertation, it seems that there is an optimum way to allocate resouces for measuring X and Y in ANOCO for each combination of values of σ^2_e , σ^2_ε , and σ^2_{xy} . The changes in $\underline{a_0}$ as a function of changes in these three parameters seem reasonable. The next section seeks verification of these results through Monte Carlo procedures. This verification seems necessary since the theoretical results assume large sample size. The Monte Carlo results are based on small size but meet all other assumptions of ANOCO.

METHOD AND PROCEDURE OF THE MONTE CARLO INVESTIGATION

The Monte Carlo investigation was carried out to check on the allocation strategy derived in the previous section. Some values of σ_ε^2, σ_e^2, and σ_{xy}^2 were selected for the investigation. They are .1, .5, and .9 and c = 100. There are therefore, twenty-seven sets of values resulting from all the possible combinations of σ_ε^2, σ_e^2, and σ_{xy}^2. The values of $\underline{a_0}$, computed from the allocation formula for each set of data are displayed in Table 4.

The twenty-seven sets of values of X and Y with the specified values of σ_ε^2, σ_e^2, and σ_{xy}^2 were generated through the use of a computer program.[1] For each set of values, 20 values of X and Y were randomly generated for data analysis. Details of the steps of data generation and data analysis are shown in Table 5.

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[1]The author would like to thank Dr. W.J. Kennedy for his substantial contribution to this research.

Table 4. Optimum allocation, a_0 for each of twenty-seven sets of parameters σ^2_{xy}, σ^2_e, and σ^2_ε, c = 100

Set of parameters				Set of parameters			
σ^2_{xy}	σ^2_e	σ^2_ε	a_0	σ^2_{xy}	σ^2_e	σ^2_ε	a_0
.1	.1	.1	76	.5	.5	.9	35
.1	.1	.5	52	.5	.9	.1	93
.1	.1	.9	28	.5	.9	.5	82
.1	.5	.1	91	.5	.9	.9	64
.1	.5	.5	59	.9	.1	.1	51
.1	.5	.9	56	.9	.1	.5	26
.1	.9	.1	97	.9	.1	.9	11
.1	.9	.5	91	.9	.5	.1	76
.1	.9	.9	83	.9	.5	.5	91
.5	.1	.1	59	.9	.5	.9	28
.5	.1	.5	32	.9	.9	.1	91
.5	.1	.9	15	.9	.9	.5	77
.5	.5	.1	81	.9	.9	.9	56
.5	.5	.5	52				

Table 5. Outline of program for data generation and data analysis

Step	Operation
1	Take a sample of size 20 from a population of x, $y \sim N(0,1,\rho^2)$, with three specified ρ^2,.1,.5, and .9.
2	Split the sample into two groups of 10.
3	Add 10 to y values of one group, i.e., treatment effect.
4	Sample 10 values from $N(0,1)$, i.e., e values.
5	Compute $Y = \sqrt{A_1}y + \sqrt{A_2}e$ where the pair of values, A_1 and A_2, take the following values; .9 and .1, .5 and .5, .1 and .9. (At this step we have $Y \sim N(0,1)$ with $\sigma_e^2 = .1, .5,$ and .9).
6	Sample 10 values from $N(0,1)$, i.e., ε values.
7	Compute $X = \sqrt{B_1}x + \sqrt{B_2}\varepsilon$ where the pair of values, B_1 and B_2, take the following values; .9 and .1, .5 and .5, .1 and .9. (At this step we have X $N(0,1)$ with $\sigma_\varepsilon^2 = .1,.5,$ and .9).
8	Repeat Step 4-5 $\underline{a}$ times and average.
9	Repeat Step 6-7 100-$\underline{a}$ times and average.
10	Use Y (Step 8) and X (Step 9) values to compute ANOCO.
11	Repeat Step 1-10 up to 100 times and average
12	Repeat Step 1-11 for every value of a: $\underline{a_0}$, certain values above $\underline{a_0}$, certain values below $\underline{a_0}$, and at $\underline{a}$=50.

RESULTS AND DISCUSSIONS

The empirical values of T (see page 23) at different allocations for each set of parameters are displayed in Tables 6-32 along with the corresponding theoretical T values. In order to help in a clearer examination of the results, graphic presentations of T values of each table are shown in Figures 11-37.

Examination of these graphs along with the corresponding tables shows differences in the level of the profiles of the empirical and theoretical T values. These differences, as discussed next, have no crucial effect on the success or utility of this optimum allocation strategy. The reason for the differences was due to using different corrections for df (constant) terms in the derivation of the allocation formula. The derivation in the second section was

$$F = \frac{2n-3}{2(n-1) + t_X^2} \; \frac{t_Y^2 - 2rt_Xt_Y + r^2t_X^2}{1-r^2}$$

. As sample size, n is small (n=10 in this investigation) $F \neq \dfrac{t_Y^2 - r^2}{1-r^2} = 1+nK_\alpha^2T$. For such small sample sizes $E(t_X^2) = \frac{n-1}{n-2} \neq 1$. This results in

$$E(F) \simeq \frac{n-2}{n-1}\left[\frac{\sigma_Y^2+\sigma_e^2 - \dfrac{\sigma_{XY}^2}{\sigma_X^2+\sigma_\varepsilon^2}\;\dfrac{n-3}{n-2}}{\sigma_Y^2+\sigma_e^2 - \dfrac{\sigma_{XY}^2}{\sigma_X^2+\sigma_\varepsilon^2}} + nK_\alpha^2T\right].$$

It can be seen that

Table 6. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma_{xy}^2 = .1$, $\sigma_e^2 = .1$, and $\sigma_\varepsilon^2 = .1$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	12	1.08	1.30	24.09	5.93	1
	50	1.11	1.08	21.94	4.39	2
	68	1.11	1.03	21.40	4.50	2
	75	1.11	1.15(1.14)[a]	22.76(47.22)	3.97(3.11)	3.4[b]
$a_0 =$	76	1.11	1.02	21.33	4.46	2
	83	1.11	1.24(1.16)	24.20(47.73)	4.29(3.67)	3.4
	84	1.11	1.02	21.35	4.33	2
	91	1.11	1.16(1.13)	23.07(46.89)	2.87(3.39)	3.4
	93	1.11	1.32	24.22	5.51	1

[a]Values in parentheses are from 4.

[b]4 is a sample of size 40.

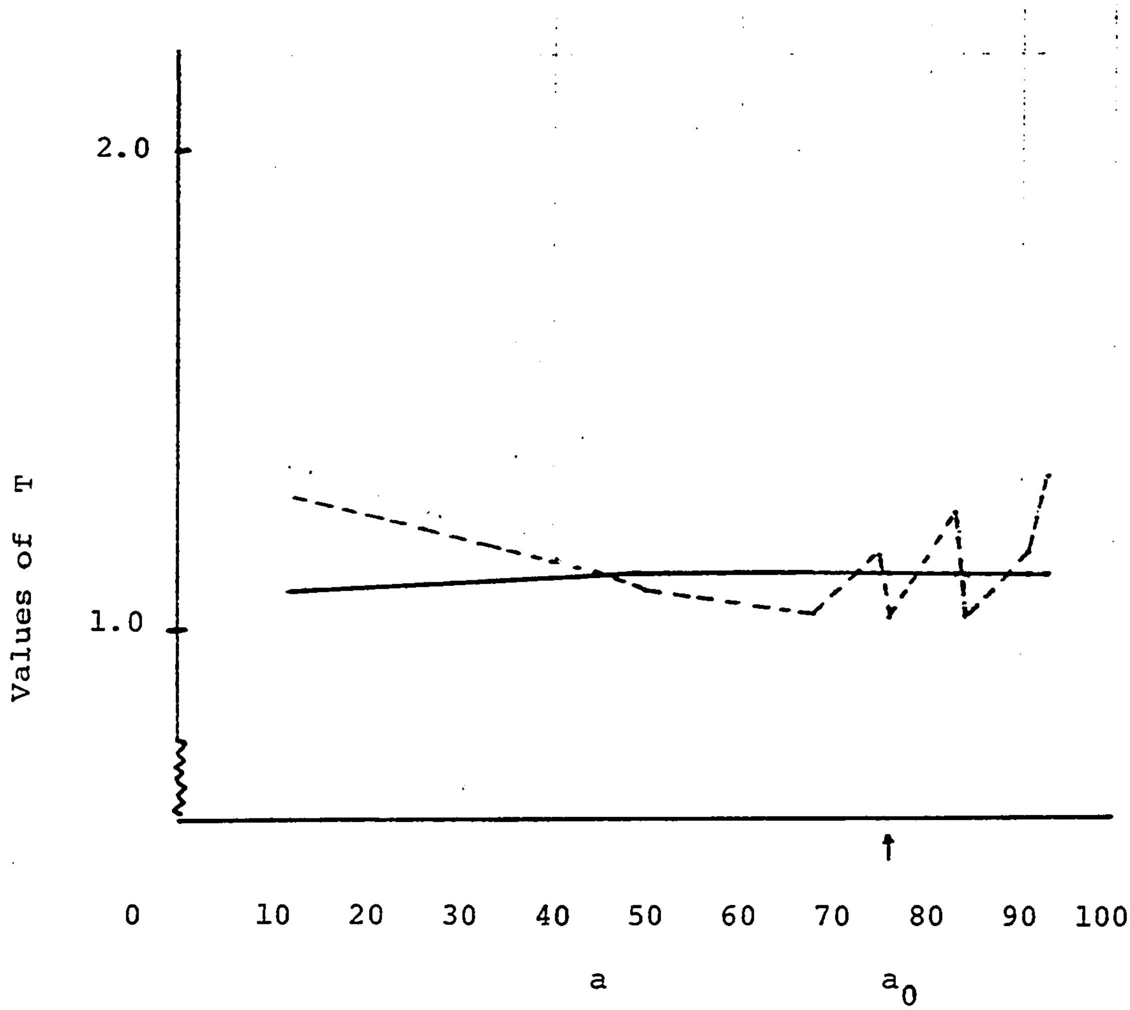


Figure 11. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ_{xy}^2=.1, σ_e^2=.1, and σ_ε^2=.1 $\underline{a_0}$ is expected optimum allocation.

Table 7. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .1$, $\sigma^2_e = .1$, and $\sigma^2_\varepsilon = .5$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	12	1.10	1.37	24.72	4.96	1
	47	1.11	1.25	23.56	3.58	2
	50	1.11	1.21	23.19	4.07	2
$a_0 =$	52	1.11	1.26	23.71	4.34	2
	57	1.11	1.26	23.73	4.13	2
	88	1.10	1.36	24.63	5.09	1

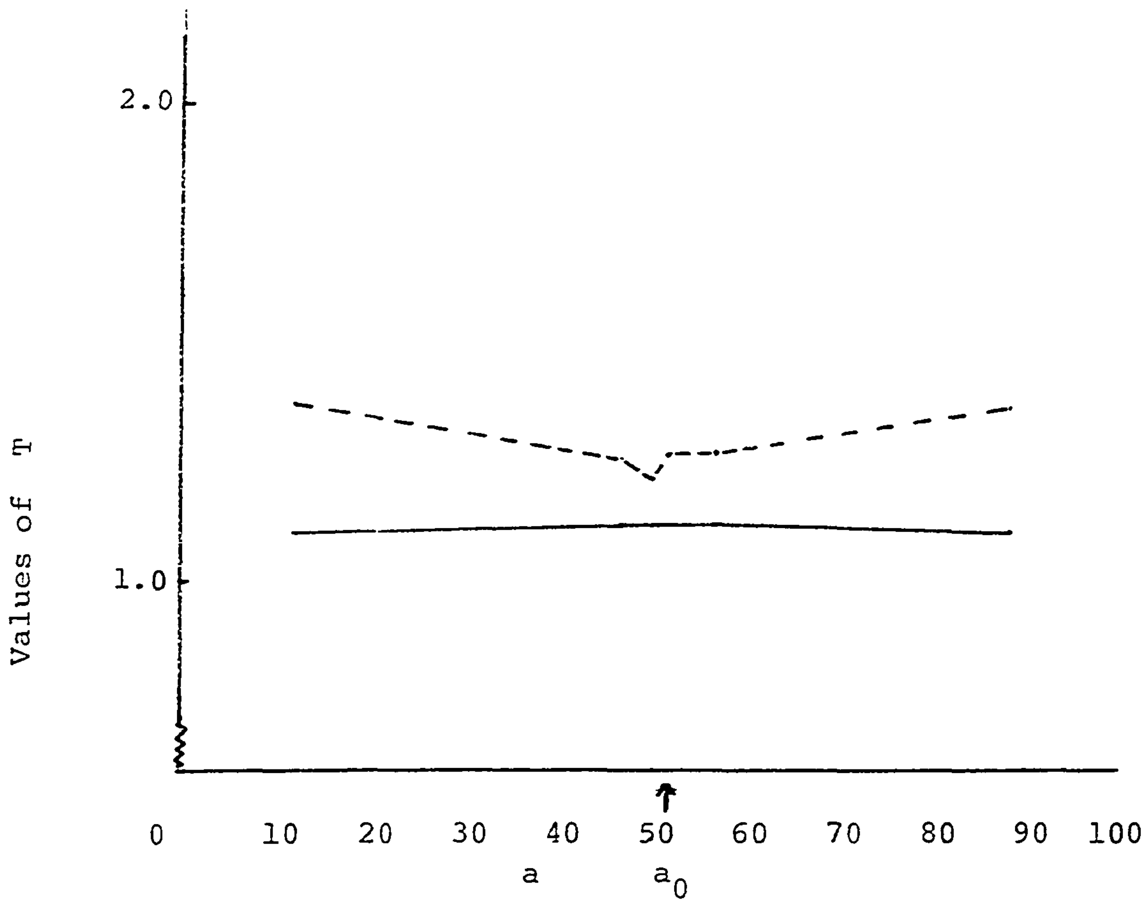


Figure 12. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.1, σ^2_e=.1, and σ^2_ε=.5 $\underline{a_0}$ is expected optimum allocation.

Table 8. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .1$, $\sigma^2_e = .1$, and $\sigma^2_\varepsilon = .9$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	11	1.09	1.41	25.06	8.51	1
	25	1.05	1.11	22.21	4.85	2
$a_0 =$	28	1.09	1.16	22.72	4.91	2
	31	1.09	1.12	22.31	4.83	2
	50	1.09	1.13	22.39	4.87	2
	76	1.08	1.24	23.49	5.48	1
	88	1.06	1.11(1.13)[a]	21.84(46.94)	3.35(4.28)	3.4[b]
	97	1.03	1.30(1.15)	25.26(47.40)	4.53(3.79)	3.4

[a]Values in parentheses are from 4.

[b]4 is a sample of size 40.

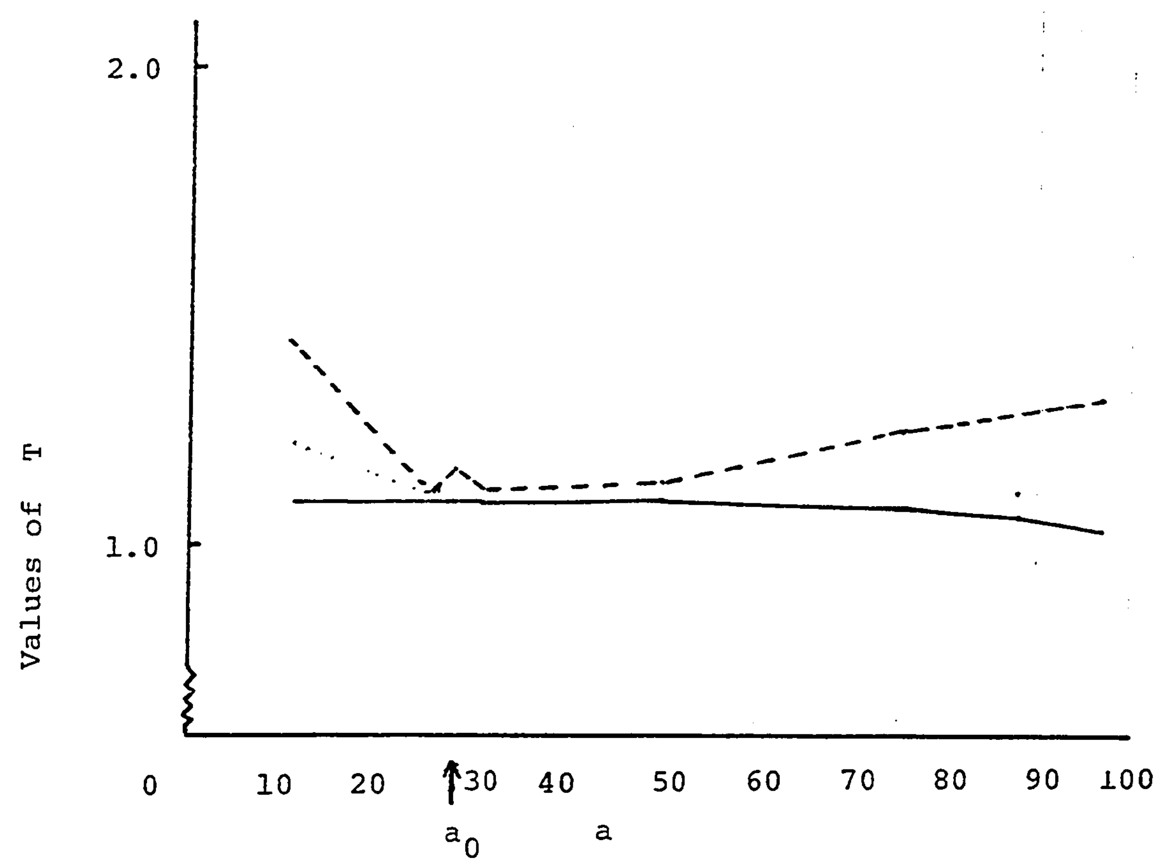


Figure 13. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.1, σ^2_e=.1, and σ^2_ε=.9 $\underline{a_0}$ is expected optimum allocation.

Table 9. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .1$, $\sigma^2_e = .5$, and $\sigma^2_\varepsilon = .1$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	12	1.03	1.09	22.03	3.75	1
	50	1.09	1.38	24.81	3.23	2
	82	1.09	1.38	24.80	2.91	2
a_0 =	91	1.09	1.41	25.01	3.11	2
	97	1.09	1.21	23.17	4.05	1
	98	1.09	1.39	24.90	3.08	2

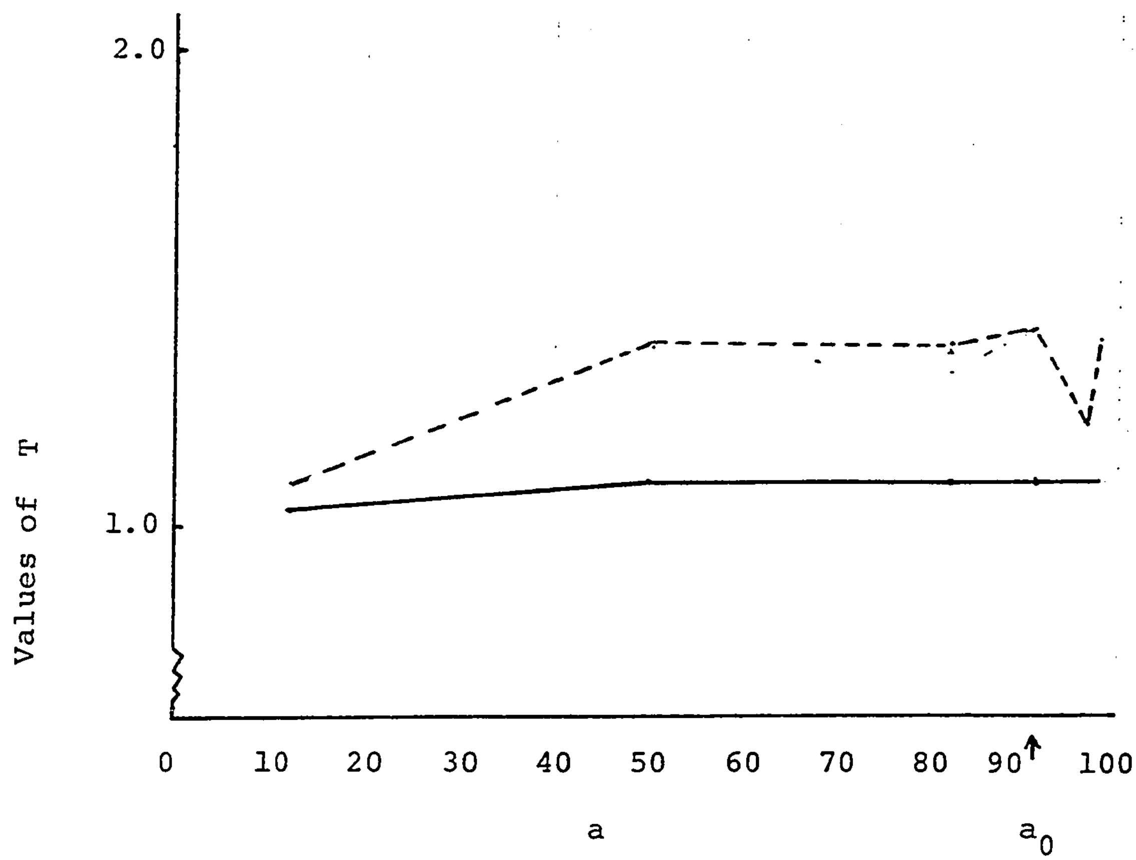


Figure 14. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when $\sigma_{xy}^2=.1$, $\sigma_e^2=.5$, and $\sigma_\varepsilon^2=.1$ $\underline{a_0}$ is expected optimum allocation

Table 10. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma_{xy}^2 = .1$, $\sigma_e^2 = .5$, and $\sigma_\varepsilon^2 = .5$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	17	1.04	1.08	21.95	5.96	1
	50	1.09	1.44	25.33	4.95	2
	69	1.09	1.37	24.73	4.24	2
$a_0 =$	77	1.09	1.42	25.12	4.20	2
	85	1.09	1.39	24.82	4.40	2
	96	1.07	1.14	22.56	4.19	1

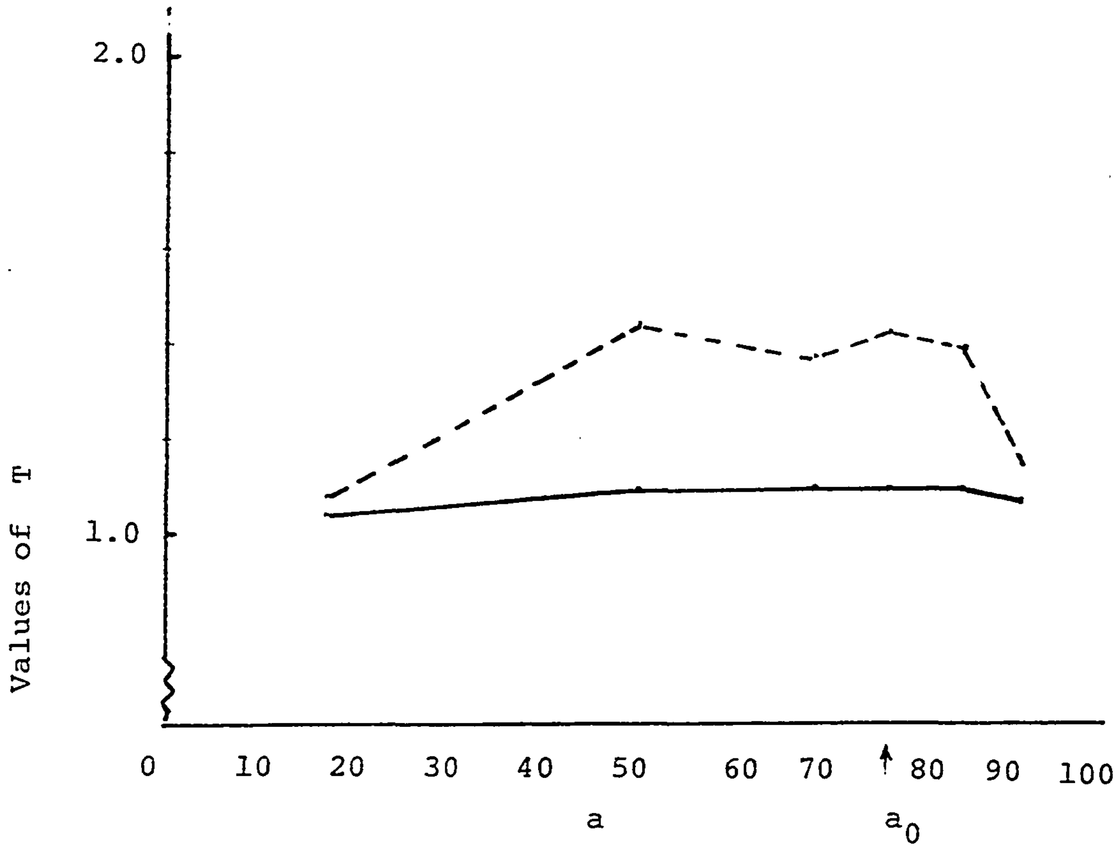


Figure 15. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.1, σ^2_e=.5, and σ^2_ε=.5 $\underline{a_0}$ is expected optimum allocation

Table 11. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ af different allocations for the set of data with parameters $\sigma^2_{xy} = .1$, $\sigma^2_e = .5$, and $\sigma^2_\varepsilon = .9$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	15	1.02	1.21	23.23	5.13	1
	50	1.07	1.21	23.18	4.33	2
a_0 =	56	1.07	1.22	23.27	4.18	2
	61	1.07	1.27	23.74	4.58	2
	72	1.06	1.24	23.49	5.76	2
	94	1.02	1.24	23.44	5.20	1

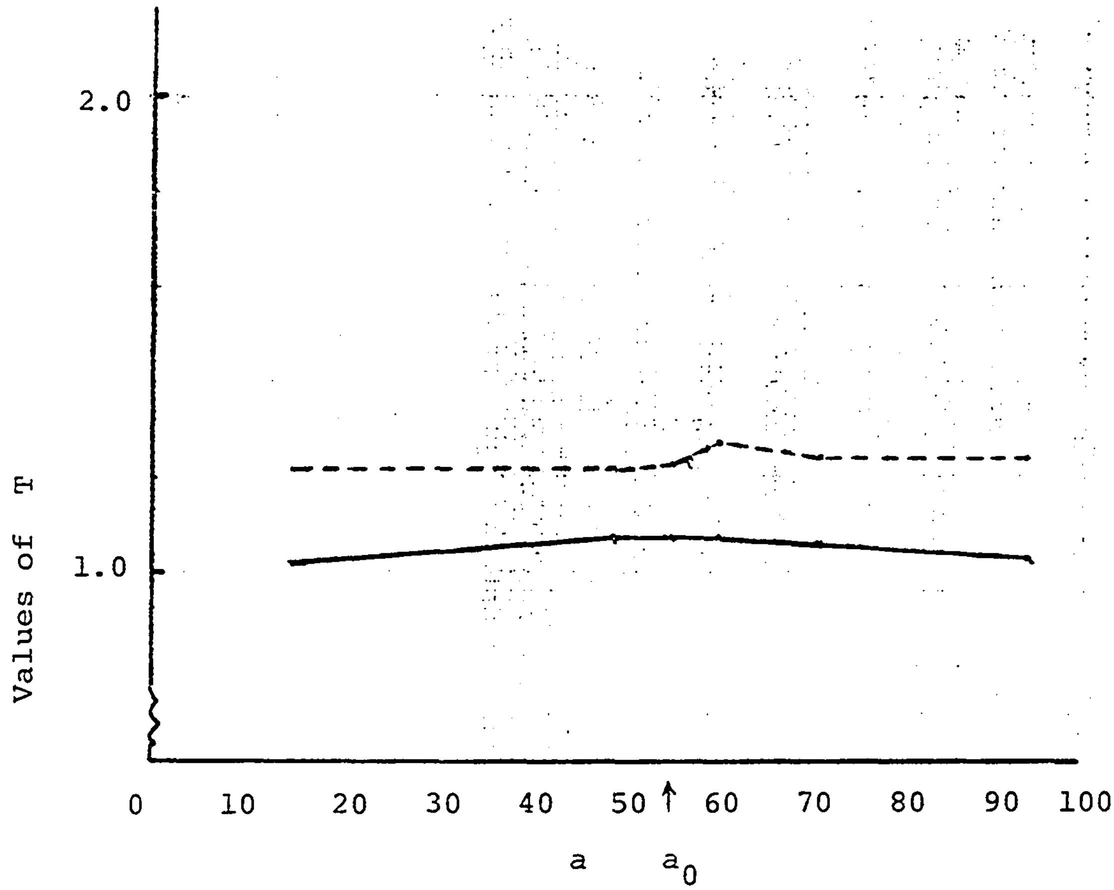


Figure 16. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.1, σ^2_e=.5, and σ^2_ε=.9 $\underline{a_0}$ is expected optimum allocation

Table 12. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .1$, $\sigma^2_e = .9$, and $\sigma^2_\varepsilon = .1$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	24	0.78	0.93	20.35	3.49	1
	25	0.79	0.85(0.88)[a]	19.01(41.55)	3.09(3.42)	3.4[b]
	31	0.84	0.99(0.81)	20.39(39.65)	3.78(3.26)	3.4
	50	0.93	1.22(0.94)	21.56(42.76)	5.27(3.36)	3.4
	88	0.99	1.23	22.31	4.47	2
				23.31	4.53	3
a_0 =	97	1.00	1.24	23.42	4.62	2
	98	1.00	1.13	21.95(22.94)	2.94(4.45)	1.2

[a]Values in parentheses are from 4.

[b]4 is a sample of size 40.

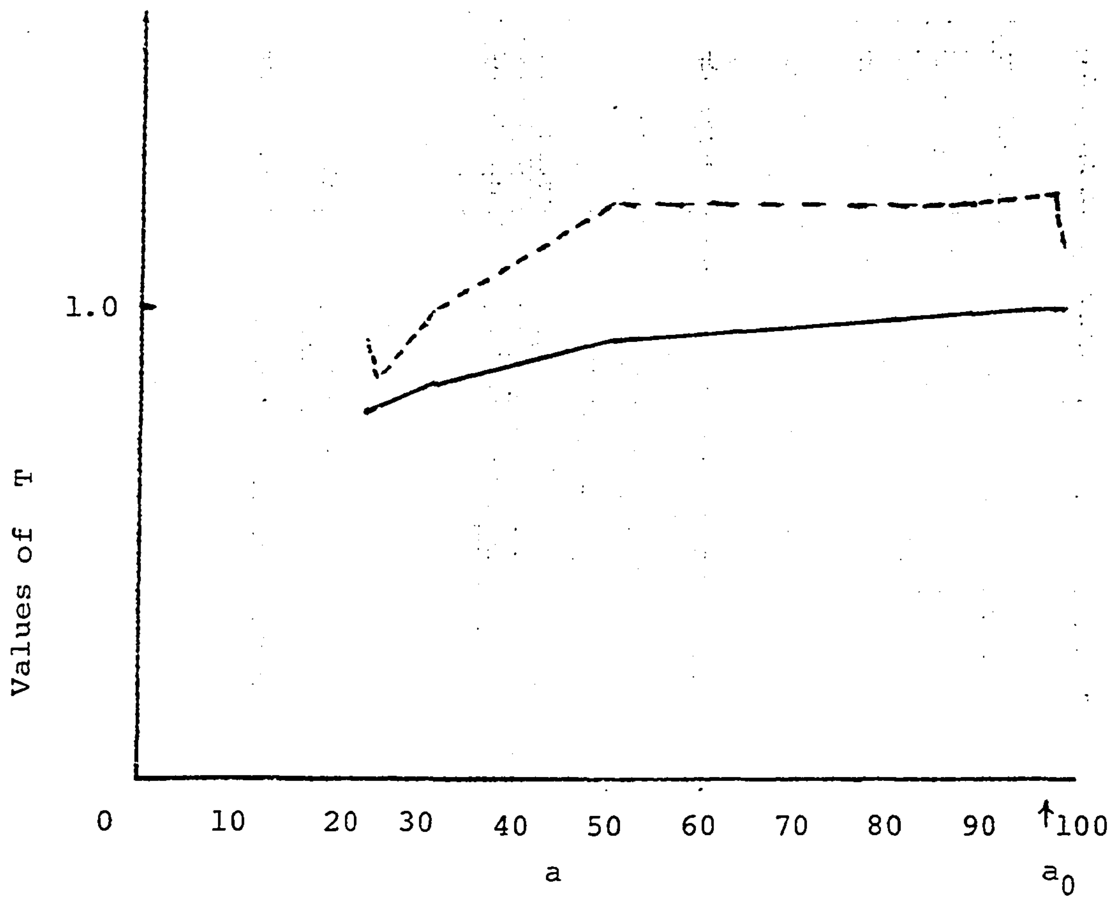


Figure 17. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when $\sigma_{xy}^2=.1$, $\sigma_e^2=.9$, and $\sigma_\varepsilon^2=.1$ $\underline{a_0}$ is expected optimum allocation

Table 13. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .1$, $\sigma^2_e = .9$, and $\sigma^2_\varepsilon = .5$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	23	0.77	0.83	19.19	3.43	1
	50	0.92	1.05	21.60	3.47	2
	82	0.99	1.31	24.12	2.88	2
a_0 =	91	0.99	1.40	24.98	4.98	2
	98	0.98	1.23	20.44(24.57)	2.55(5.27)	1.2

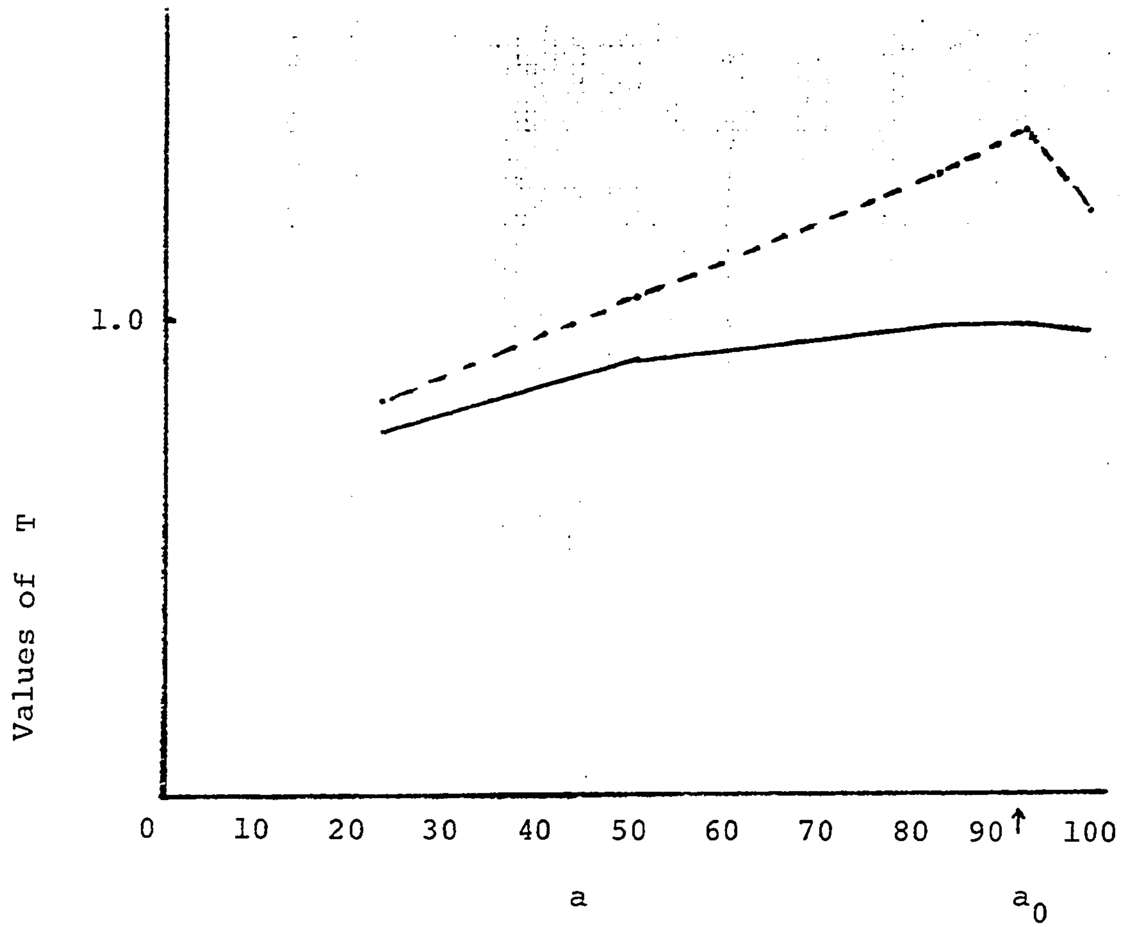


Figure 18. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.1, σ^2_e=.9, and σ^2_ε=.5 $\underline{a_0}$ is expected optimum allocation

Table 14. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma_{xy}^2 = .1$, $\sigma_e^2 = .9$, and $\sigma_\varepsilon^2 = .9$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	23	0.77	0.80	18.91	4.69	1
	50	0.91	1.23	23.38	4.80	2
	68	0.95	1.08(1.04)[a]	22.37(45.01)	3.61(4.11)	3.4[b]
	75	0.96	1.14	22.57	5.41	2
	76	0.96	1.01(0.98)	20.98(43.75)	3.96(3.21)	3.4
$a_0 =$	83	0.96	1.15	22.58	5.66	2
	84	0.96	1.07(1.03)	22.73(44.86)	4.53(4.73)	3.4
	91	0.95	1.11	22.25	4.02	2
	97	0.94	1.03	21.43	4.70	1

[a]Values in parentheses are from 4.

[b]4 is a sample of size 40.

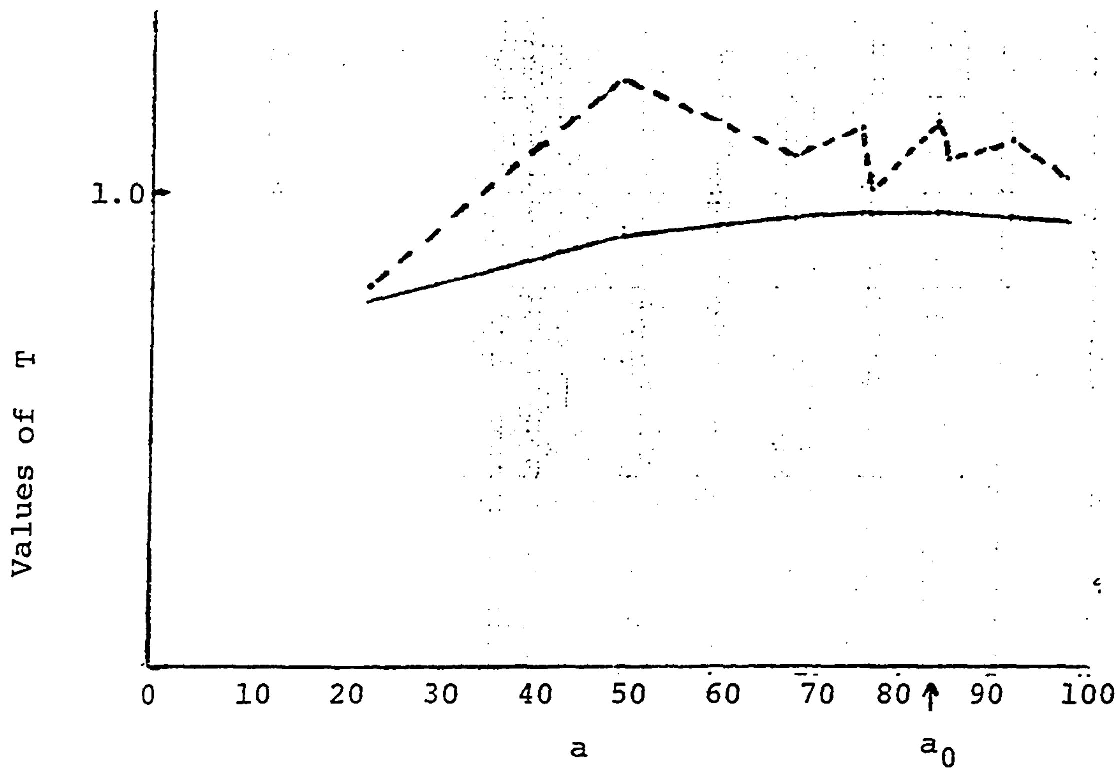


Figure 19. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.1, σ^2_e=.9, and σ^2_ε=.9 $\underline{a_0}$ is expected optimum allocation

Table 15. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .5$, $\sigma^2_e = .1$, and $\sigma^2_\varepsilon = .1$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	13	1.96	2.07	30.36	4.17	1
	50	1.99	1.89	29.02	4.61	2
	53	1.99	1.91	29.17	4.99	2
$a_0 =$	59	1.99	1.93	29.25	4.95	2
	65	1.99	1.88	28.95	4.66	2
	91	1.97	2.06	30.26	4.83	1

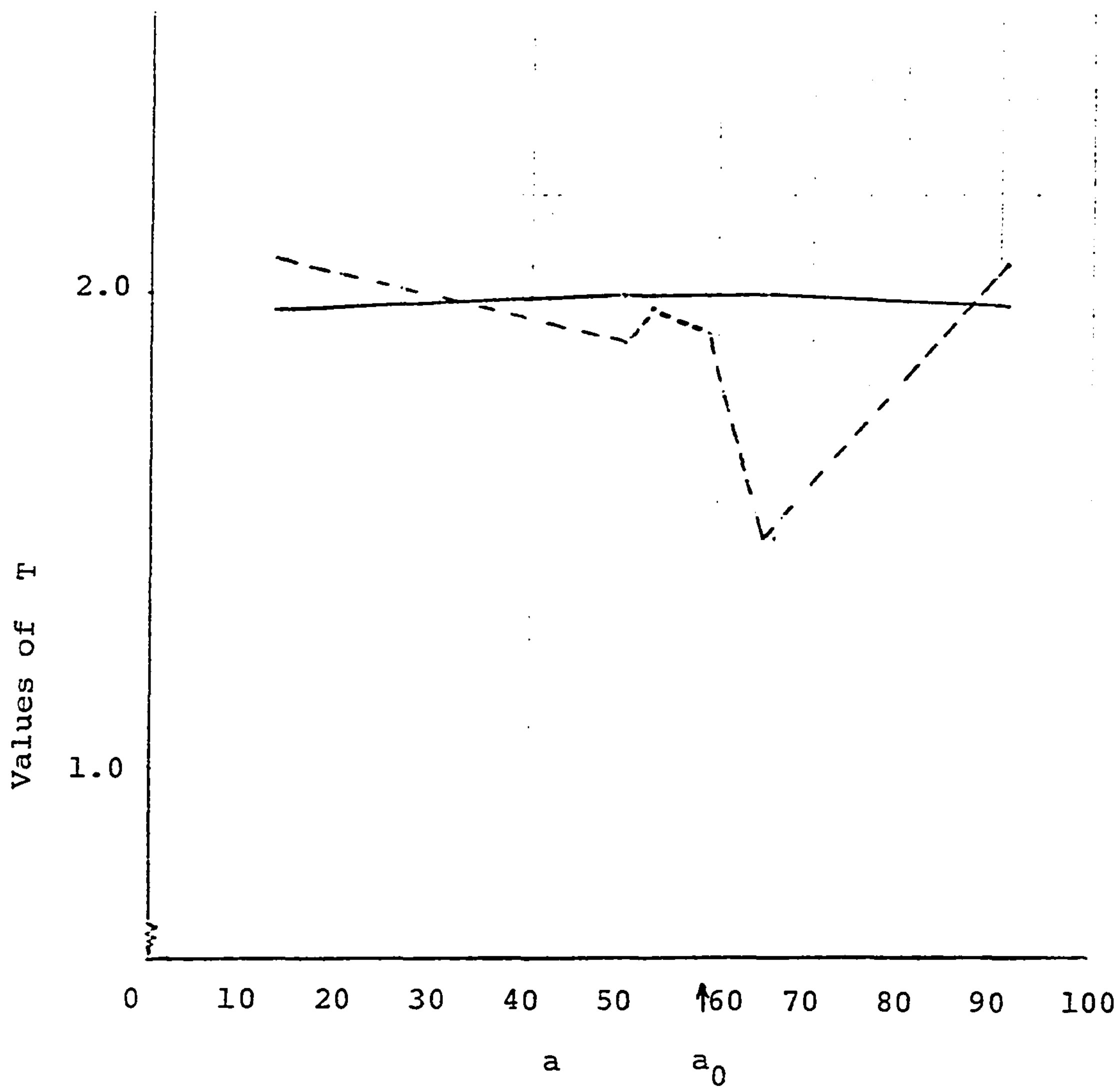


Figure 20. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.5, σ^2_e=.1, and σ^2_ε=.1 $\underline{a_0}$ is expected optimum allocation

Table 16. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .5$, $\sigma^2_e = .1$, and $\sigma^2_\varepsilon = .5$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	8	1.93	2.32	32.09	4.49	1
	29	1.96	2.30	32.02	7.51	2
a_0 =	32	1.96	2.23	31.45	6.54	2
	35	1.96	2.29	31.93	7.95	2
	50	1.95	2.26	31.71	6.89	2
	86	1.87	2.22	31.46	5.44	1

Table 17. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .5$, $\sigma^2_e = .1$, and $\sigma^2_\varepsilon = .9$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	5	1.76	2.08	30.40	5.19	1
	13	1.80	2.16	30.97	6.01	2
$a_0 =$	15	1.80	2.03	30.03	5.05	2
	17	1.80	2.03	30.05	5.77	2
	50	1.71	2.06	30.30	5.09	2
	70	1.62	2.06	30.28	5.41	1

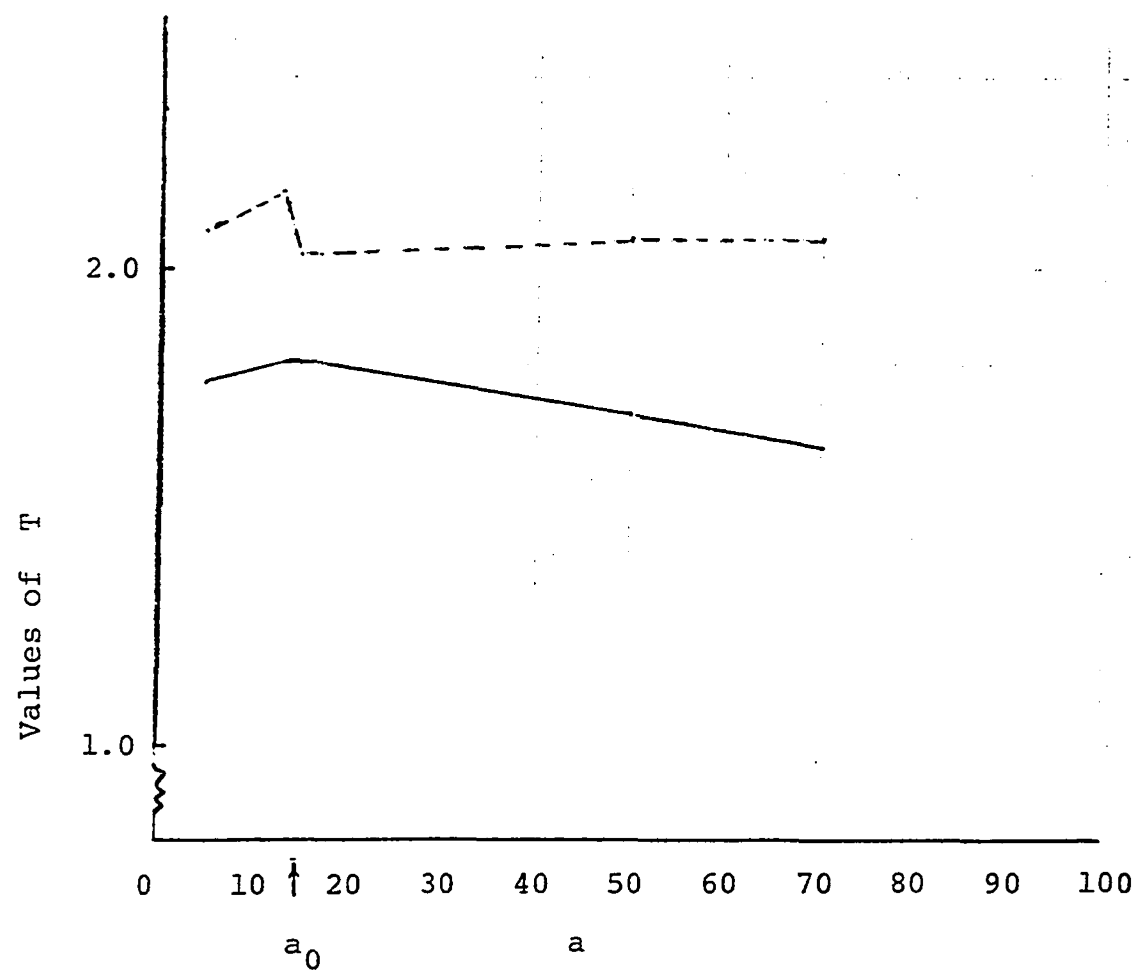


Figure 22. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when $\sigma^2_{xy} = .5$, $\sigma_e^2 = .1$, and $\sigma_\varepsilon^2 = .9$ $\underline{a_0}$ is expected optimum allocation

Table 18. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .5$, $\sigma^2_e = .5$, and $\sigma^2_\varepsilon = .1$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	15	1.76	1.85	28.71	4.58	1
	50	1.92	2.15	30.93	6.33	2
	73	1.94	2.21	31.38	5.82	2
$a_0 =$	81	1.94	2.32	32.11	6.88	2
	89	1.94	2.20	31.27	4.78	2
	95	1.92	1.92	29.27	3.96	1

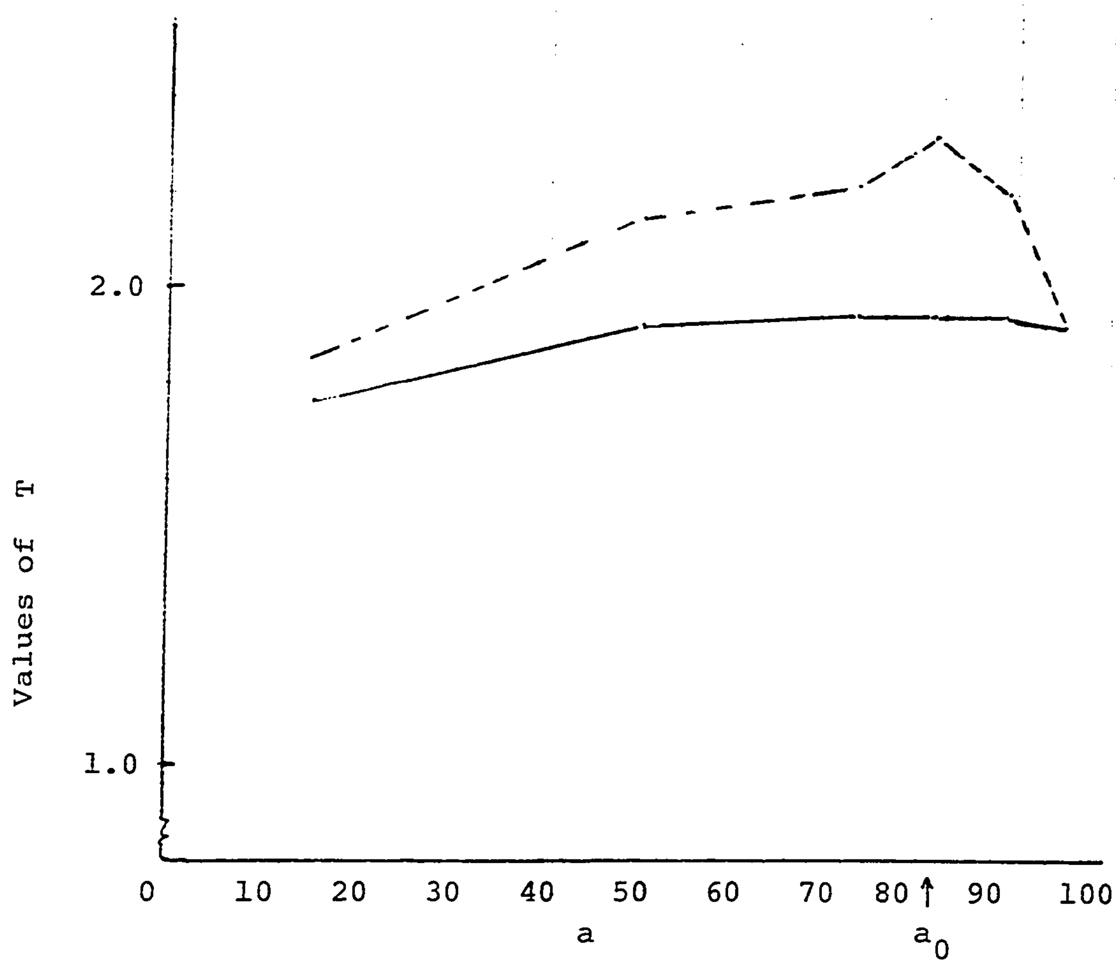


Figure 23. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when $\sigma^2_{xy}=.5$, $\sigma^2_e=.5$, and $\sigma^2_\varepsilon=.1$ $\underline{a_0}$ is expected optimum allocation

Table 19. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma_{xy}^2 = .5$, $\sigma_e^2 = .5$, and $\sigma_\varepsilon^2 = .5$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	16	1.76	2.13	30.81	4.22	1
	50	1.89	2.16	30.98	8.16	2
	53	1.89	2.19	31.21	6.64	2
a_0 =	59	1.89	2.12	30.71	7.72	2
	65	1.89	2.04	30.14	4.78	2
	90	1.79	2.07	30.34	4.22	1

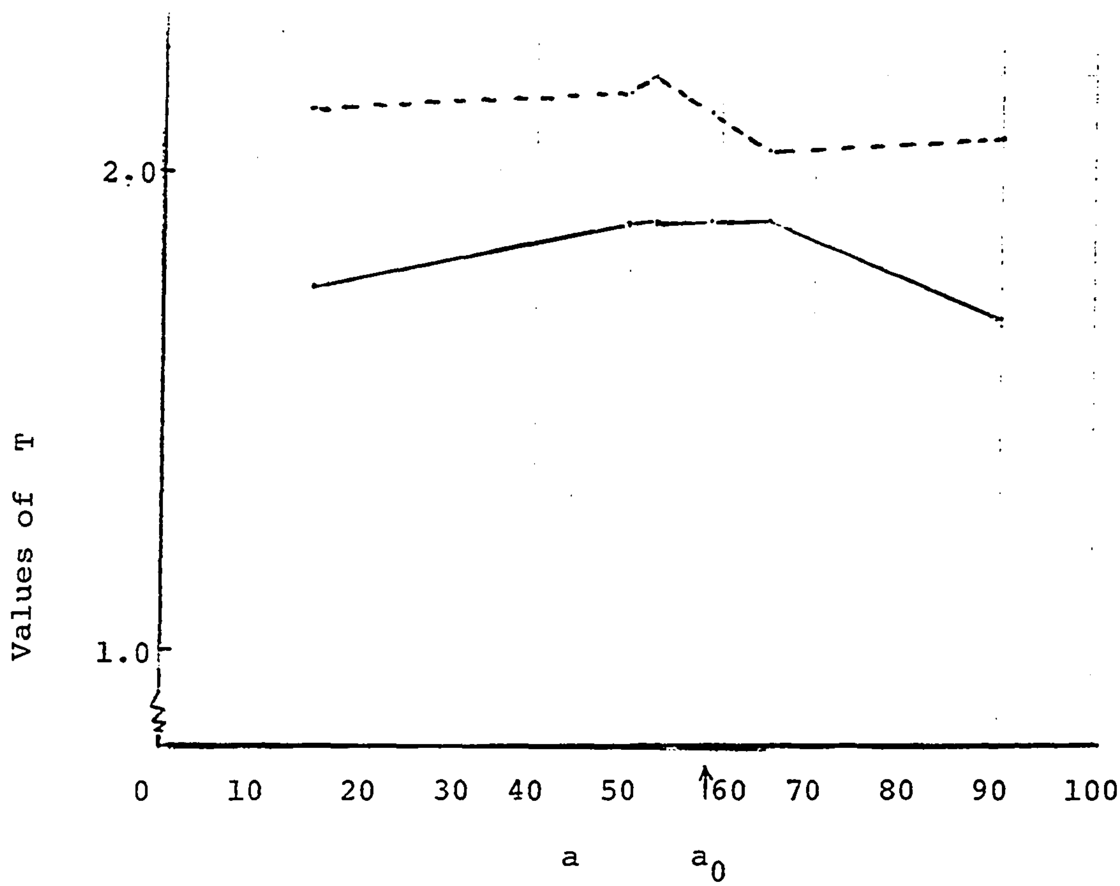


Figure 24. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.5, σ^2_e=.5, and σ^2_ε=.5 $\underline{a_0}$ is expected optimum allocation

Table 20. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .5$, $\sigma^2_e = .5$, and $\sigma^2_\varepsilon = .9$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	17	1.65	1.75	27.90	3.95	1
	31	1.69	1.99	29.18	8.68	2
a_0 =	35	1.69	1.76	27.99	5.16	2
	39	1.69	1.89	29.04	6.94	2
	50	1.68	2.04	30.16	6.47	2
	73	1.56	1.59	26.63	4.47	1

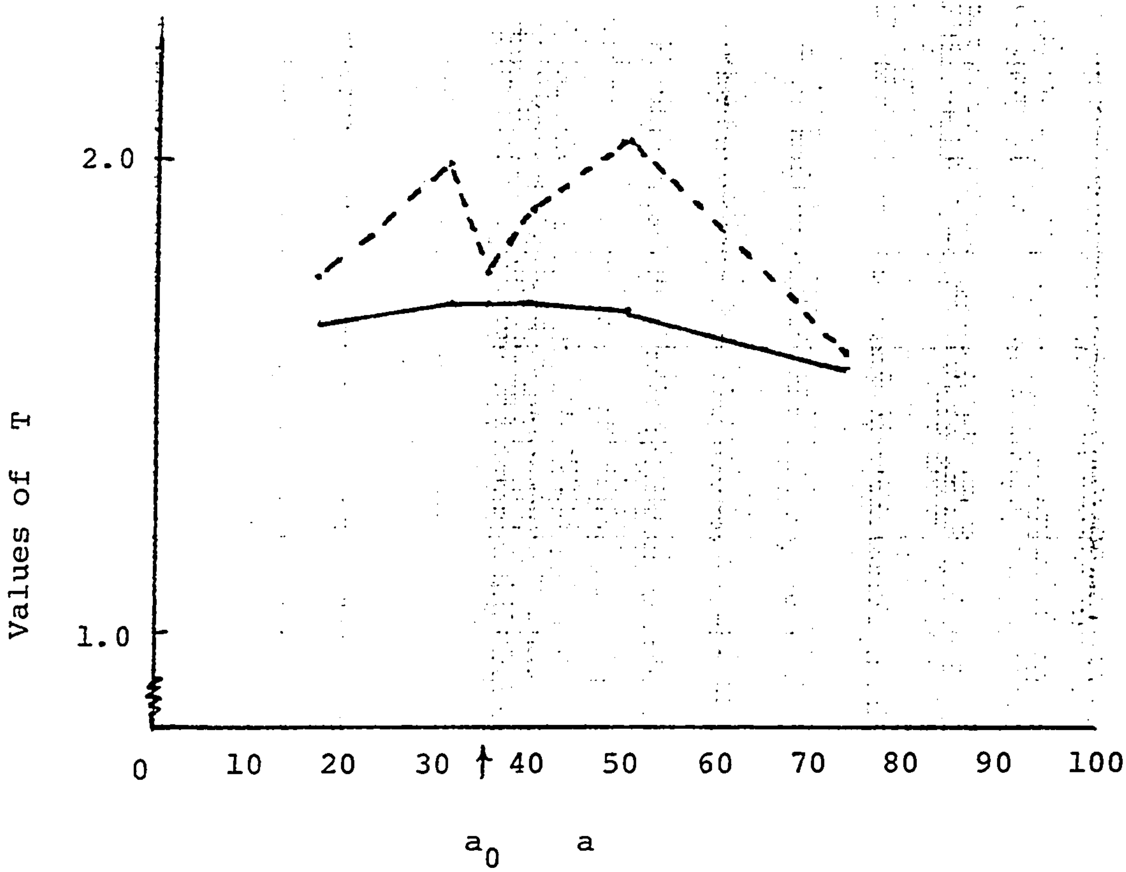


Figure 25. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.5, σ^2_e=.5, and σ^2_ε=.9 $\underline{a_0}$ is expected optimum allocation

Table 21. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .5$, $\sigma^2_e = .9$, and $\sigma^2_\varepsilon = .1$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	29	1.23	1.42	25.09	4.07	1
	50	1.47	1.84	28.52	4.65	2
	84	1.64	1.89	29.07	3.71	2
$a_0 =$	93	1.65	1.94	29.36	4.44	2
	98	1.63	1.92	29.43(28.96)	8.02(4.64)	1.2

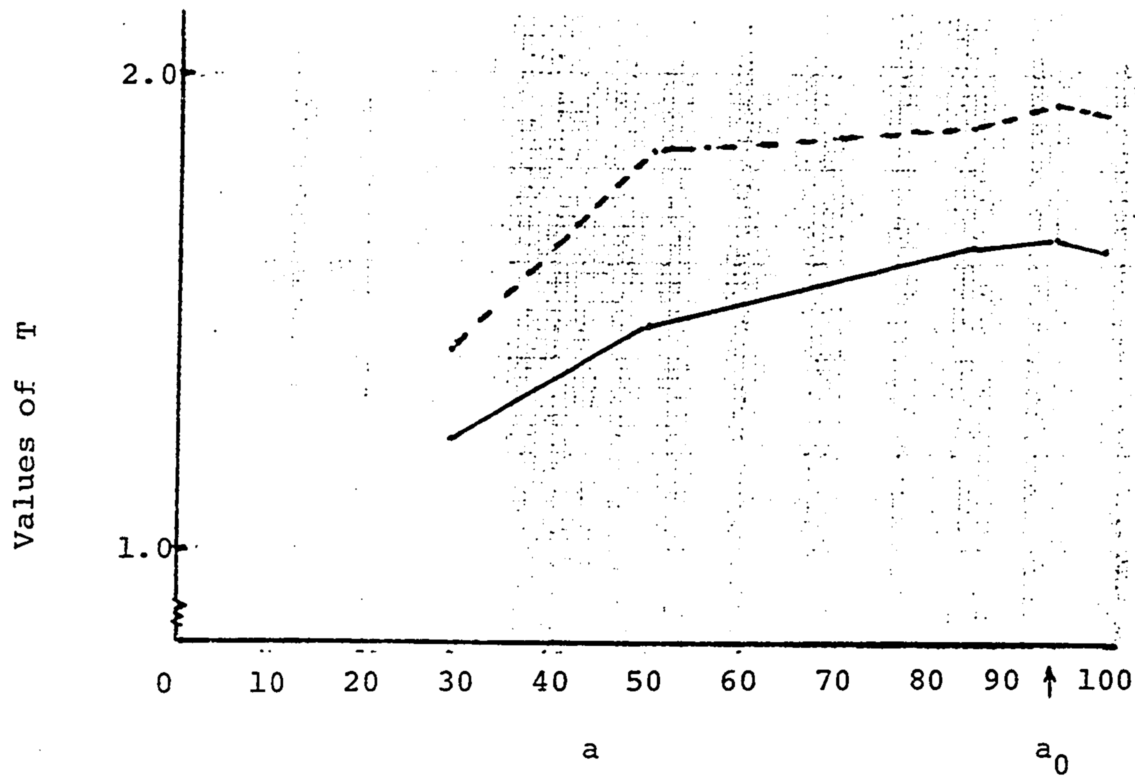


Figure 26. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.5, σ^2_e=.9, and σ^2_ε=.1 $\underline{a_0}$ is expected optimum allocation

Table 22. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .5$, $\sigma^2_e = .9$, and $\sigma^2_\varepsilon = .5$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	28	1.21	1.61	26.74	4.09	1
	50	1.45	1.66	27.17	4.99	2
	74	1.56	1.52	25.98	5.50	2
a_0 =	82	1.57	1.72	27.69	5.28	2
	90	1.55	1.79	28.26	5.97	2
	93	1.52	1.96	29.54	6.47	1

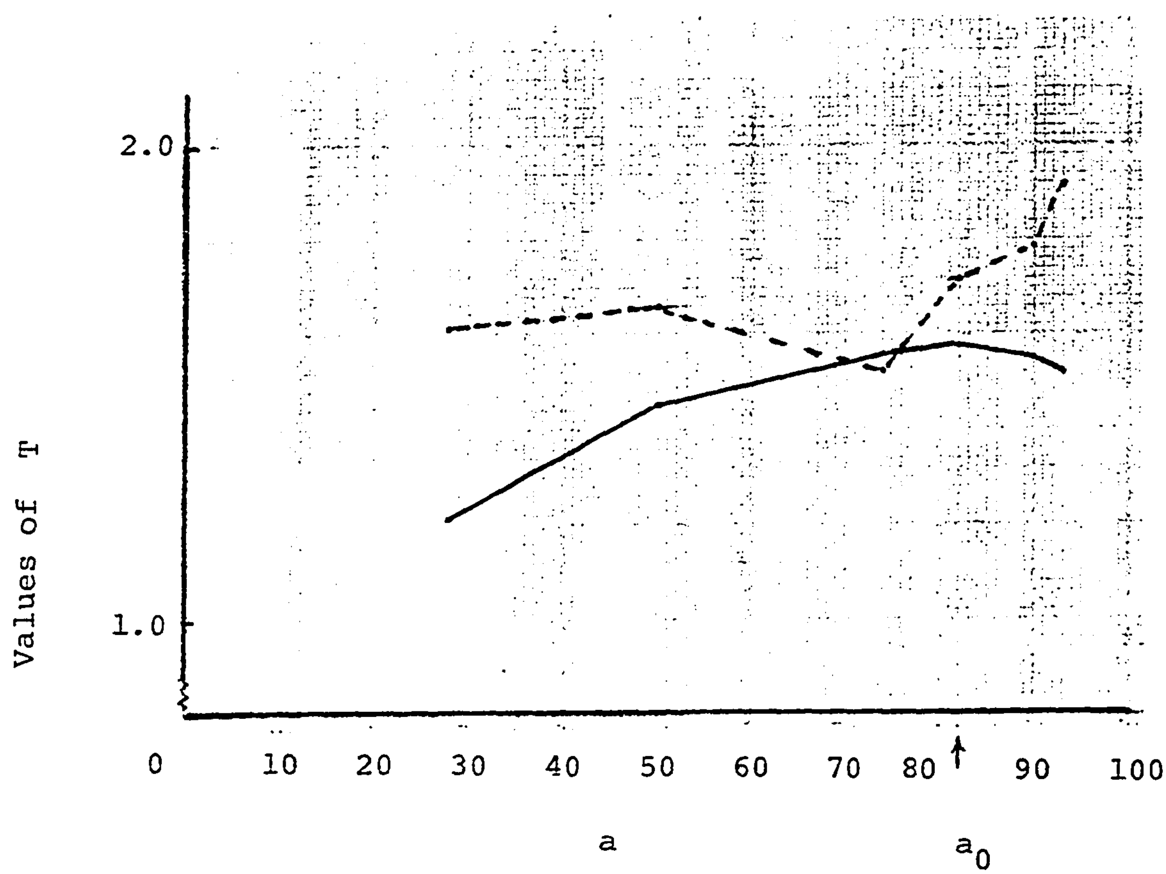


Figure 27. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.5, σ^2_e=.9, and σ^2_ε=.5 $\underline{a_0}$ is expected optimum allocation

Teble 23. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .5$, $\sigma^2_e = .9$, and $\sigma^2_\varepsilon = .9$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	31	1.18	1.47	25.60	5.28	1
	50	1.32	1.54	26.14	6.59	2
	58	1.35	1.53	26.07	5.43	2
$a_0 =$	64	1.35	1.73	27.75	5.72	2
	70	1.34	1.59	26.59	5.67	2
	88	1.22	1.51	25.90	5.22	1

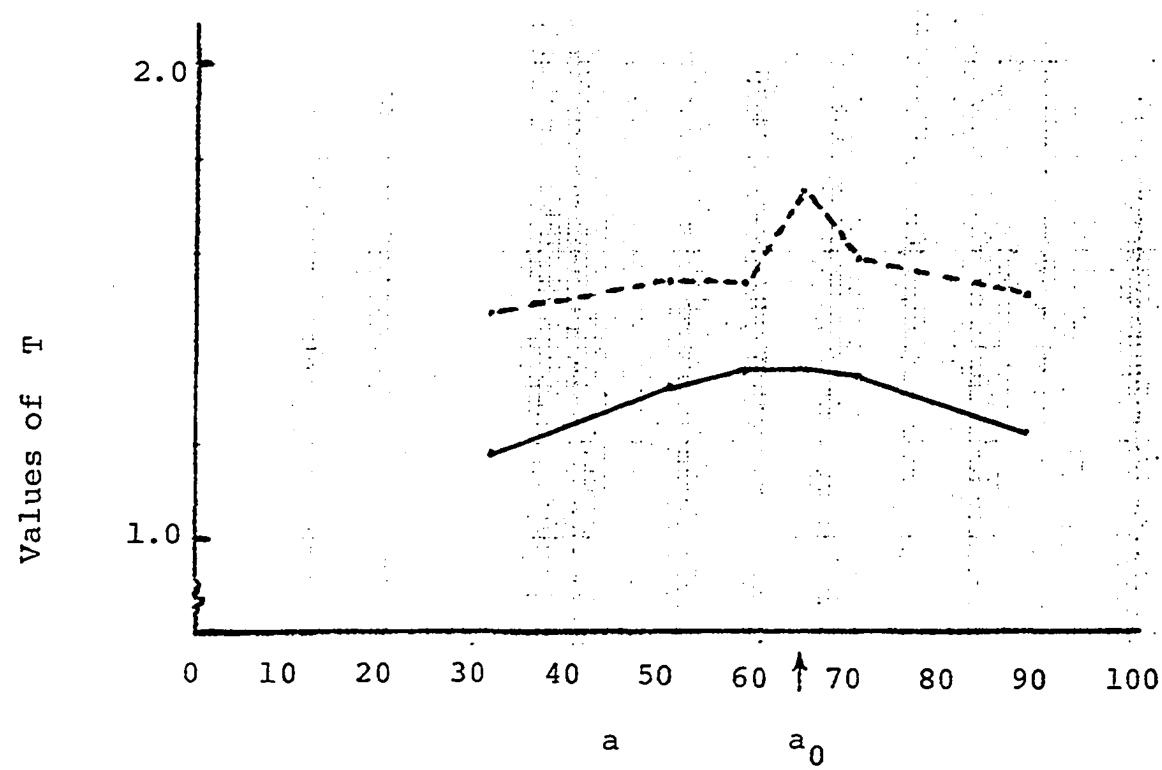


Figure 28. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.5, σ^2_e=.9, and σ^2_ε=.9 $\underline{a_0}$ is expected optimum allocation

Table 24. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .9$, $\sigma^2_e = .1$, and $\sigma^2_\varepsilon = .1$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	13	9.12	9.78	65.97	15.59	1
	46	9.59	11.20	72.09(68.97)	14.54(12.06)	2,3
	50	9.59	11.02	70.69(69.50)	15.19(10.86)	2,3
$a_0 =$	51	9.59	10.98	71.24(68.51)	15.69(9.33)	2,3
	56	9.59	10.90	70.23(68.46)	15.14(9.40)	2,3
	62	9.57	11.18	70.46	13.63	3
	86	9.23	11.28	70.83	17.37	1

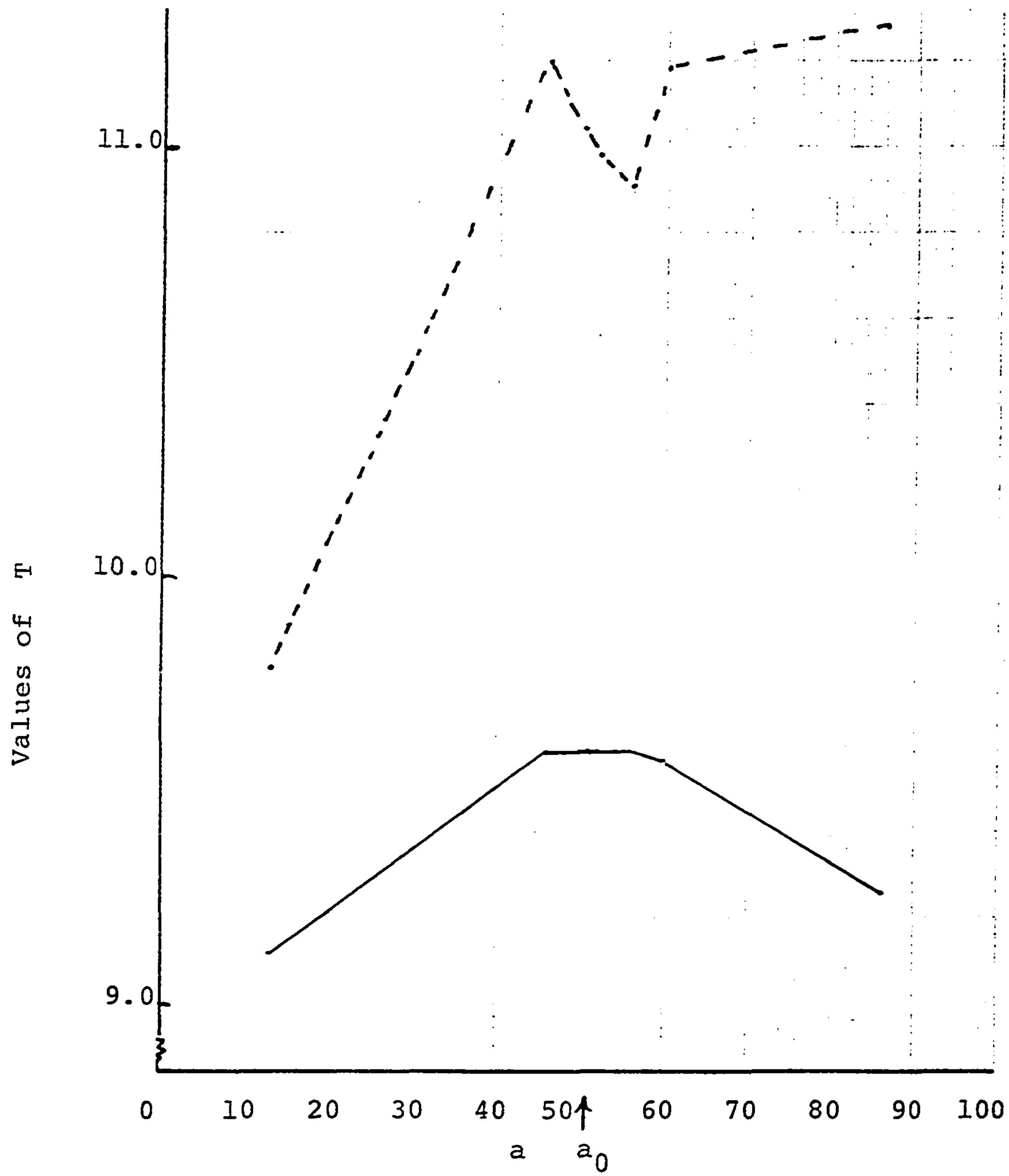
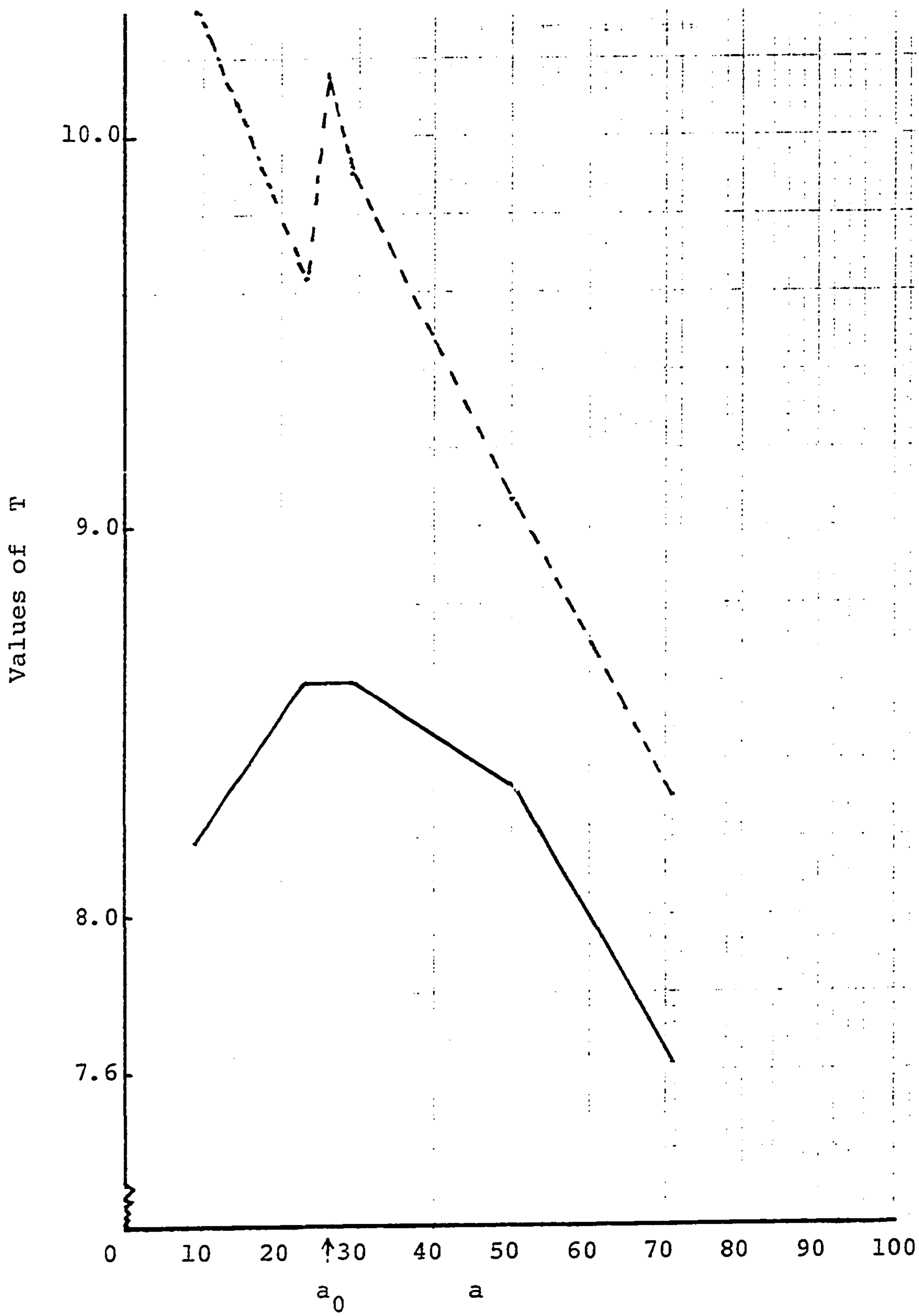


Figure 29. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when $\sigma^2_{xy}=.9$, $\sigma^2_e=.1$, and $\sigma^2_\varepsilon=.1$ $\underline{a_0}$ is expected optimum allocation

Table 25. Values of theoretical T, empirical T, empirical $\sqrt{F}$, and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .9$, $\sigma^2_e = .1$, and $\sigma^2_\varepsilon = .5$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	9	8.19	10.35	68.84	15.66	1
	23	8.60	9.73	65.78(66.86)	13.96(13.89)	2,3
a_0 =	26	8.60	10.16	67.22(65.77)	14.65(11.12)	2,3
	29	8.60	9.85	66.18(67.22)	15.17(12.51)	2,3
	50	8.33	9.07	63.51(64.69)	12.03(9.11)	2,3
	77	7.61	8.30	60.75	11.02	1

Figure 30. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when $\sigma^2_{xy}=.9$, $\sigma^2_e=.1$, and $\sigma^2_\varepsilon=.5$ $\underline{a_0}$ is expected optimum allocation

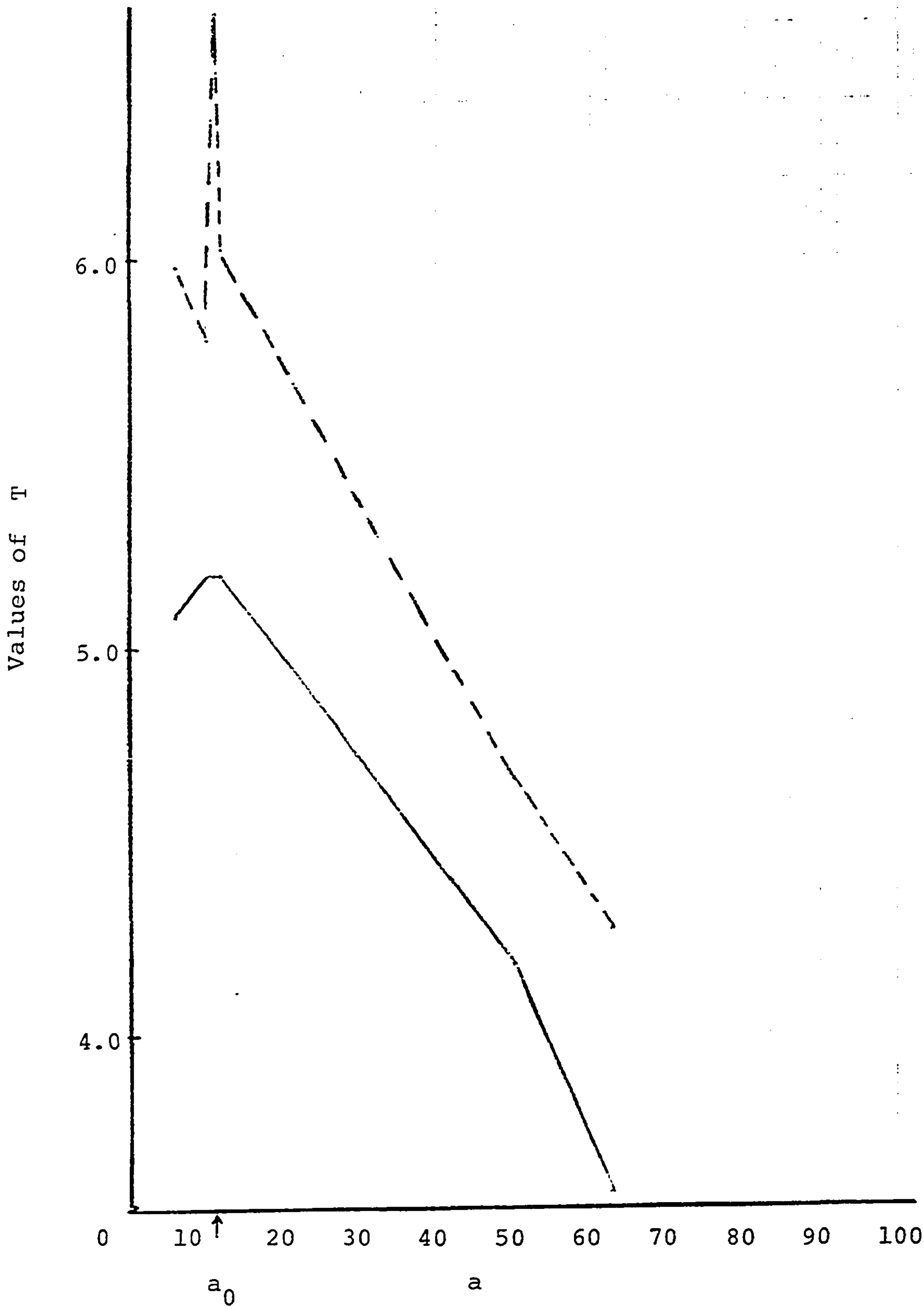


Values of T
10.0
9.0
8.0
7.6
0
10
20
30
40
50
60
70
80
90
100
a_0
a

Table 26. Values of theoretical T, empirical T, empirical $\sqrt{F}$ and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters σ^2_{xy} = ,9, σ^2_e = .1, and σ^2_ε = .9

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	6	5.08	5.98	51.58	10.17	1
	10	5.18	5.79	50.72(54.52)	9.54(7.75)	2,3
a_0 =	11	5.18	6.63	54.28(51.28)	10.92(7.13)	2,3
	12	5.18	6.02	51.72(53.23)	7.59(10.15)	2,3
	50	4.18	4.66	45.54(44.85)	9.97(6.41)	2,3
	63	3.59	4.27	43.56	6.85	1

Figure 31. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.9, σ^2_e=.1, and σ^2_ε=.9 $\underline{a_0}$ is expected optimum allocation

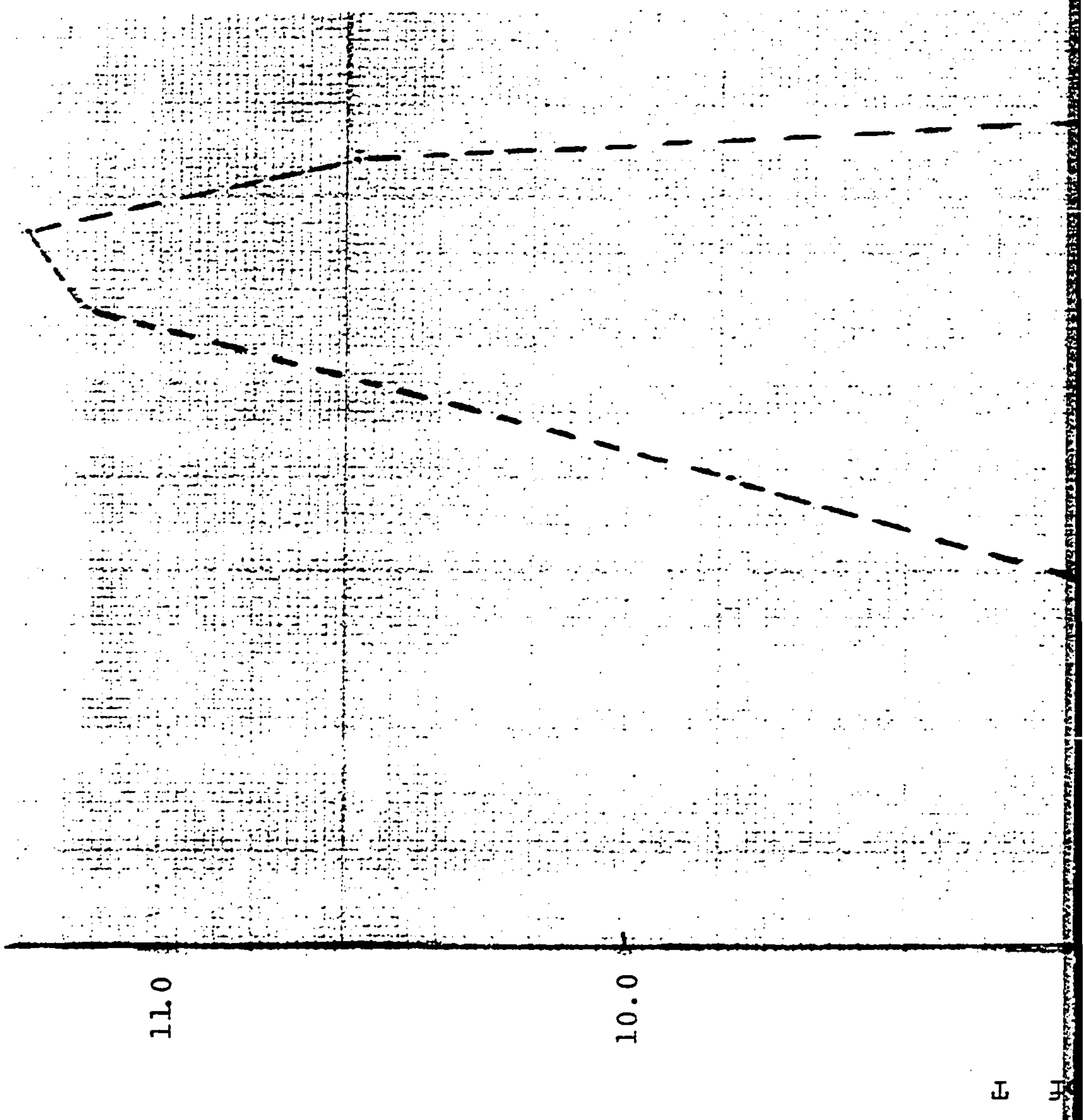


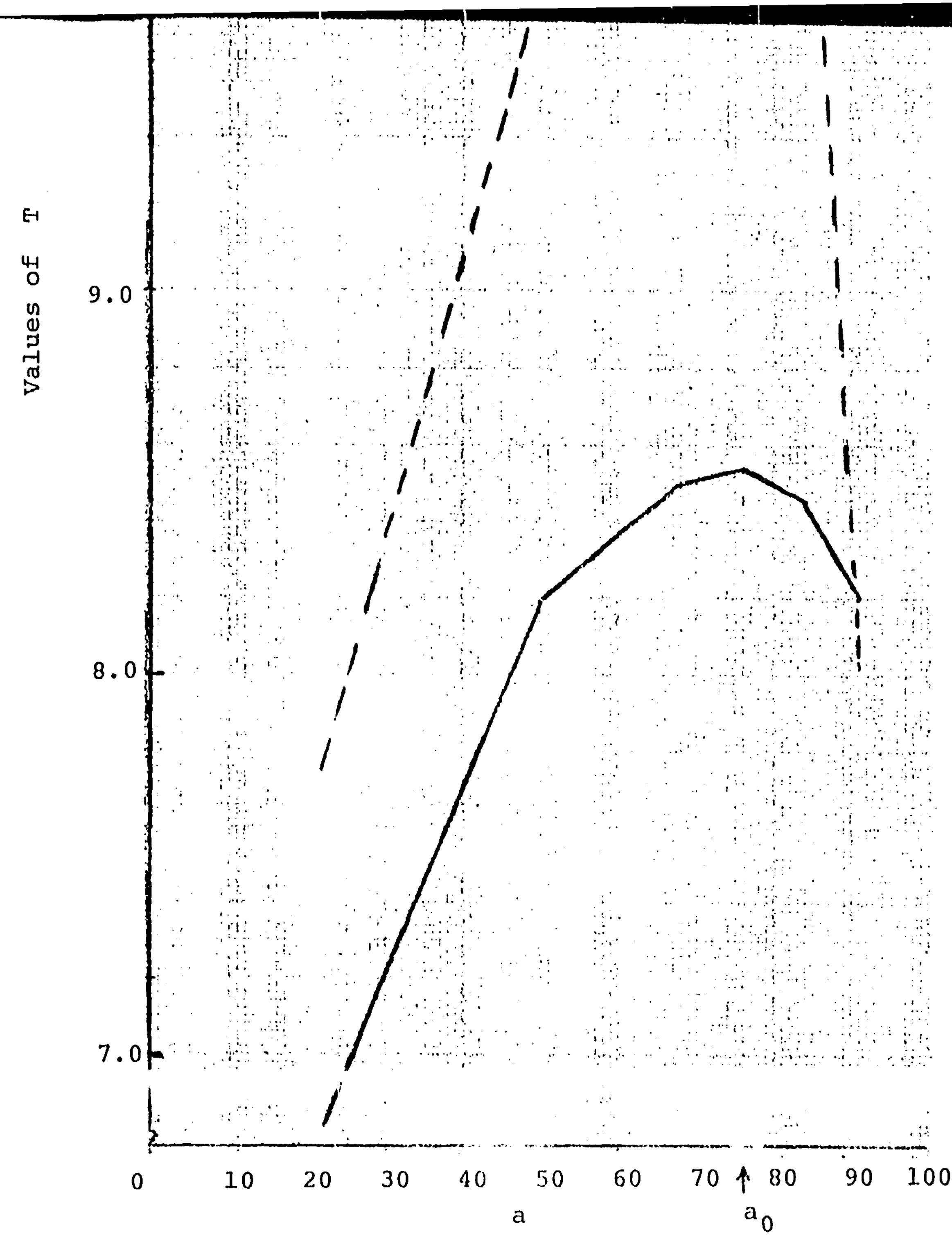
Values of T
6.0
5.0
4.0
0
10
20
30
40
50
60
70
80
90
100
a0
a

Table 27. Values of theoretical T, empirical T, empirical $\sqrt{F}$ and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .9$, $\sigma^2_e = .5$, and $\sigma^2_\varepsilon = .1$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	22	6.82	7.76	58.74	8.45	1
	50	8.19	9.78	65.93	7.05	2
	68	8.49	11.16	70.43	10.77	2
$a_0 =$	76	8.54	11.29	70.84	10.05	2
	84	8.45	10.58	68.57	9.86	2
	91	8.20	8.03	61.94	10.16	1

Figure 32. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.9, σ^2_e=.5, and σ^2_ε=.1 $\underline{a_0}$ is expected optimum allocation


11.0
10.0

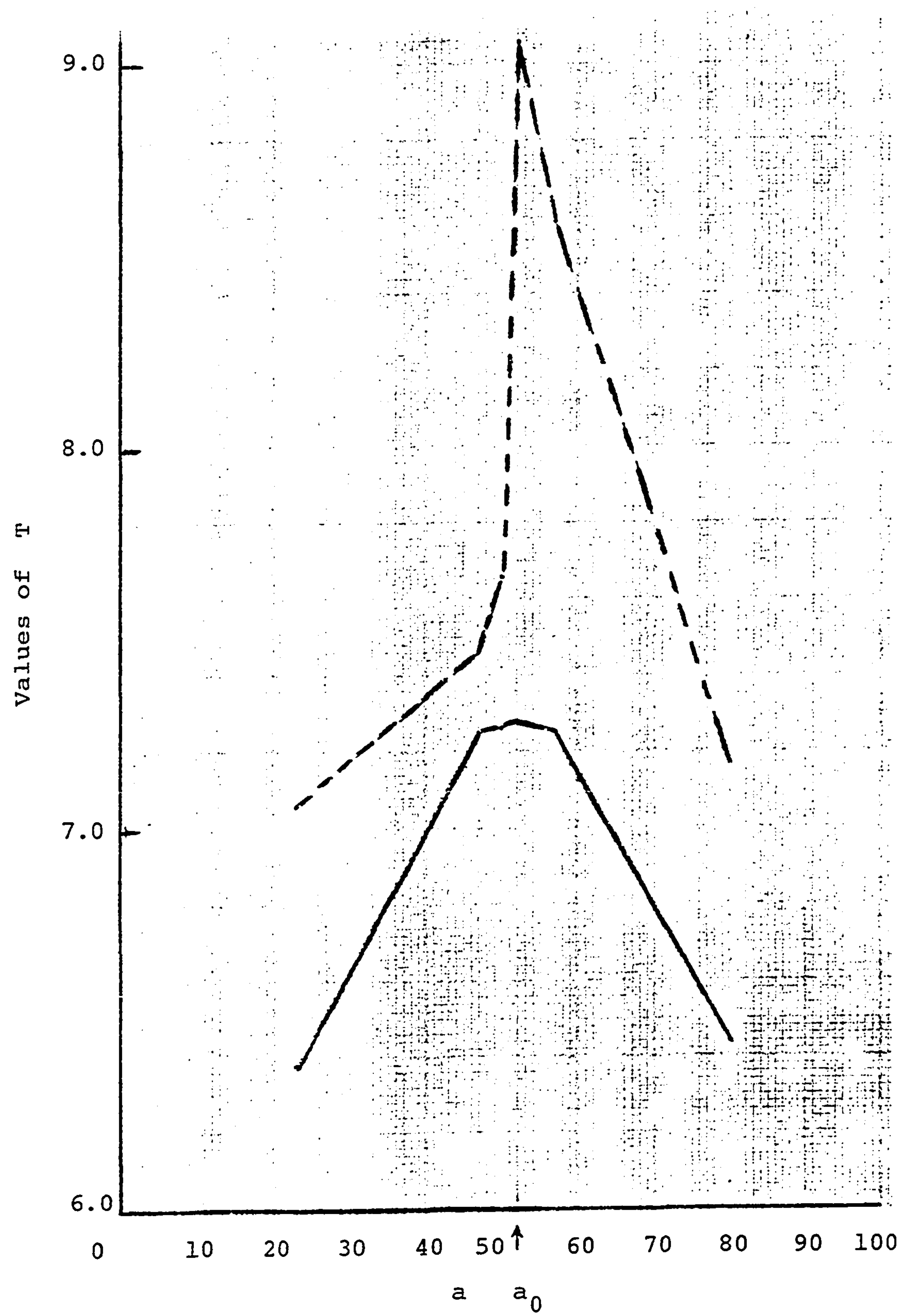


Values of T
9.0
8.0
7.0
0
10
20
30
40
50
60
70
80
90
100
a
a0

Table 28. Values of theoretical T, empirical T, empirical $\sqrt{F}$ and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .9$, $\sigma^2_e = .5$, and $\sigma^2_\varepsilon = .5$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	23	6.38	7.06	56.04	8.48	1
	47	7.25	7.46	57.58(60.84)	8.61(13.32)	2.3
	50	7.26	7.66	58.36(58.54)	11.82(10.33)	2,3
a_0 =	52	7.27	9.06	63.47(59.07)	9.10(10.91)	2,3
	87	7.25	8.59	61.78(58.05)	12.18(10.15)	2,3
	80	6.44	6.97	55.68	10.49	1

Figure 33. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.9, σ^2_e=.5, and σ^2_ε=.5 $\underline{a_0}$ is expected optimum allocation



Values of T
9.0
8.0
7.0
6.0
0
10
20
30
40
50
60
70
80
90
100
a
a_0

Table 29. Values of theoretical T, empirical T, empirical $\sqrt{F}$ and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .9$, $\sigma^2_e = .5$, and $\sigma^2_\varepsilon = .9$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	17	4.06	5.18	47.99	10.08	1
	25	4.23	5.29	48.28	10.61	2
$a_0 =$	28	4.24	4.72	45.82	7.26	2
	31	4.24	4.70	45.72	8.08	2
	50	3.89	4.19	43.14	6.69	2
	63	3.43	4.25	43.46	6.19	1

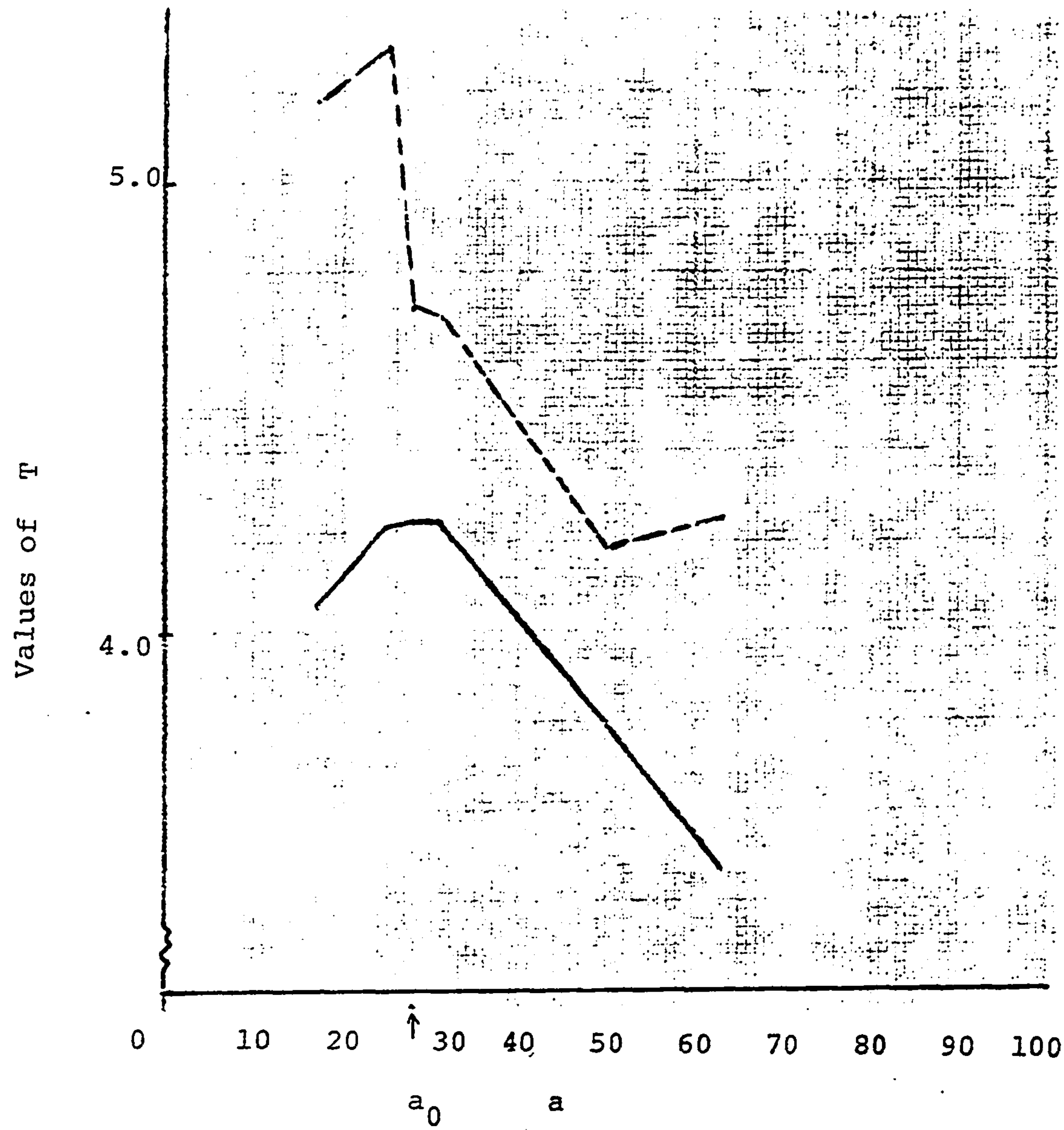
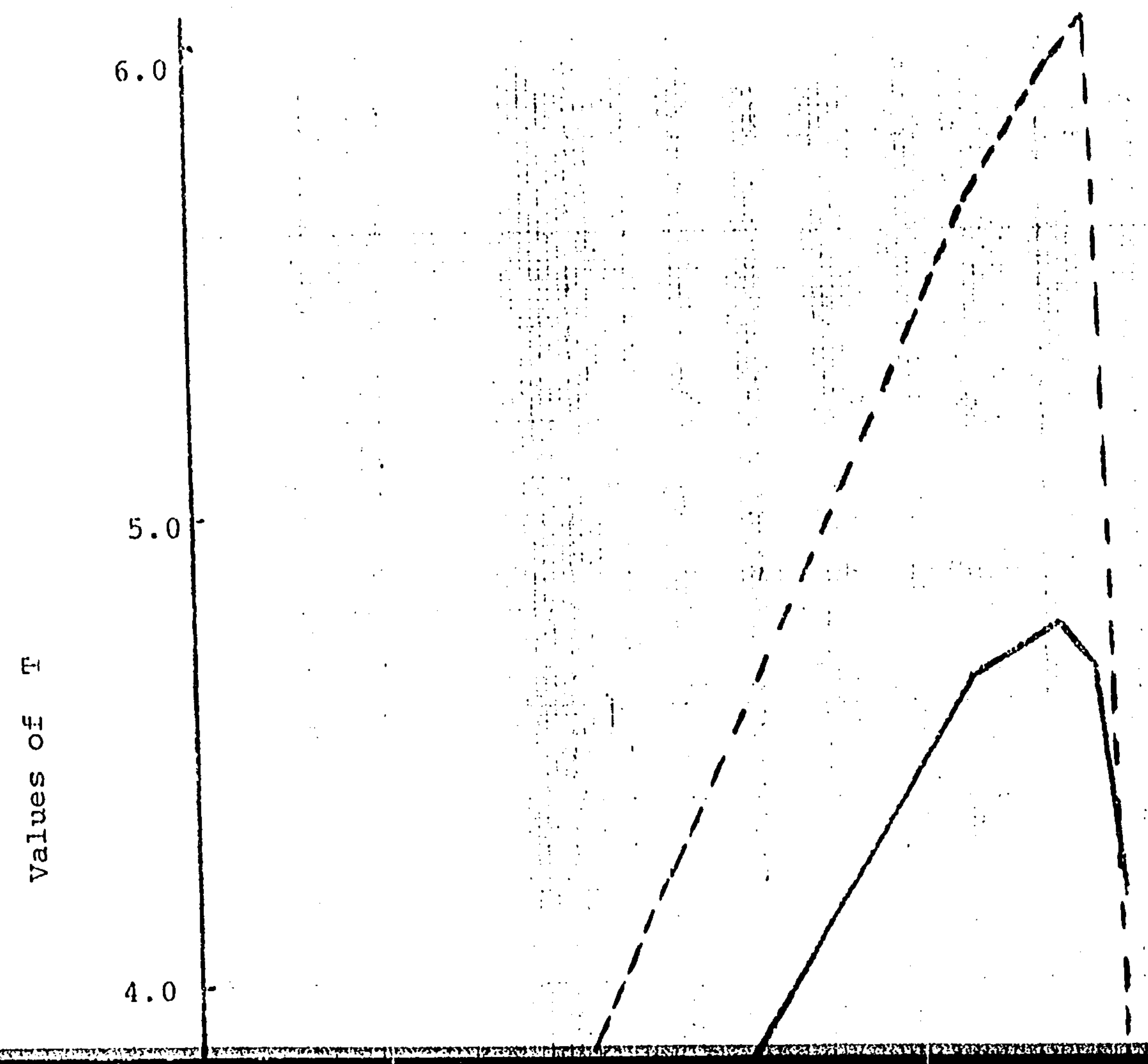


Figure 34. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when $\sigma^2_{xy}=.9$, $\sigma^2_e=.5$, and $\sigma^2_\varepsilon=.9$ $\underline{a_0}$ is expected optimum allocation

Table 30. Values of theoretical T, empirical T, empirical $\sqrt{F}$ and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .9$, $\sigma^2_e = .9$, and $\sigma^2_\varepsilon = .1$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	34	2.73	3.05	36.83	7.08	1
	50	3.55	4.24	43.42	8.50	2
	82	4.65	5.66	50.17	16.52	2
a_0 =	91	4.76	5.95	51.44	10.09	2
	95	4.67	6.04	51.83	7.21	1
	98	4.18	3.75	40.82	8.40	2

Figure 35. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when $\sigma^2_{xy}=.9$, $\sigma^2_e=.9$, and $\sigma^2_\varepsilon=.1$ $\underline{a_0}$ is expected optimum allocation


Values of T
6.0
5.0
4.0

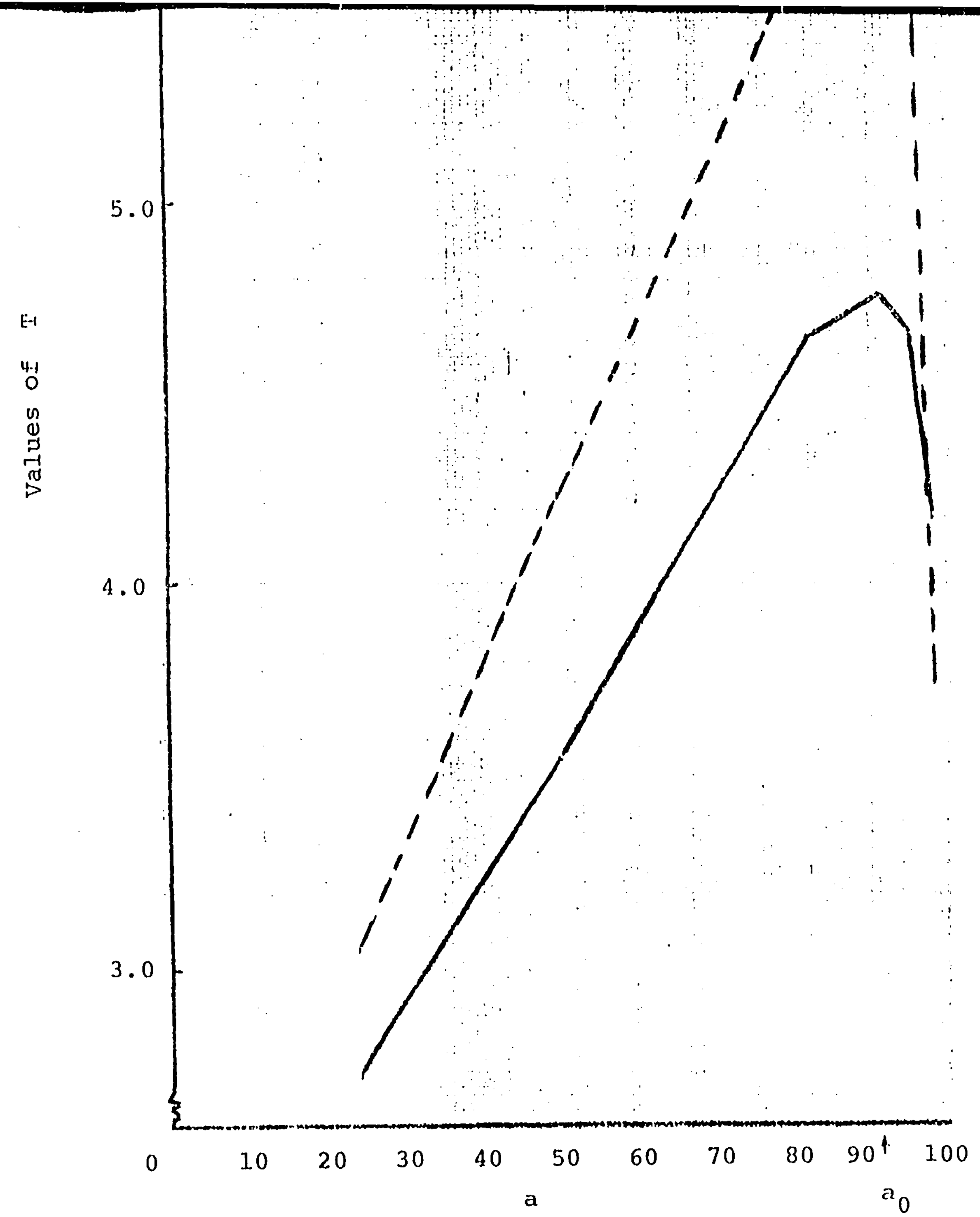

Values of T
5.0
4.0
3.0
0
10
20
30
40
50
60
70
80
90
100
a
a_0

Table 31. Values of theoretical T, empirical T, empirical $\sqrt{F}$ and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .9$, $\sigma^2_e = .9$, and $\sigma^2_\varepsilon = .5$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	37	2.79	3.22	37.85	5.82	1
	50	3.36	4.38	44.11	8.73	2
	69	3.87	4.93	46.83	7.12	2
$a_0 =$	77	3.93	4.59	45.17	7.55	2
	85	3.82	4.80	46.29	9.43	2
	88	3.68	4.33	43.86	8.99	1

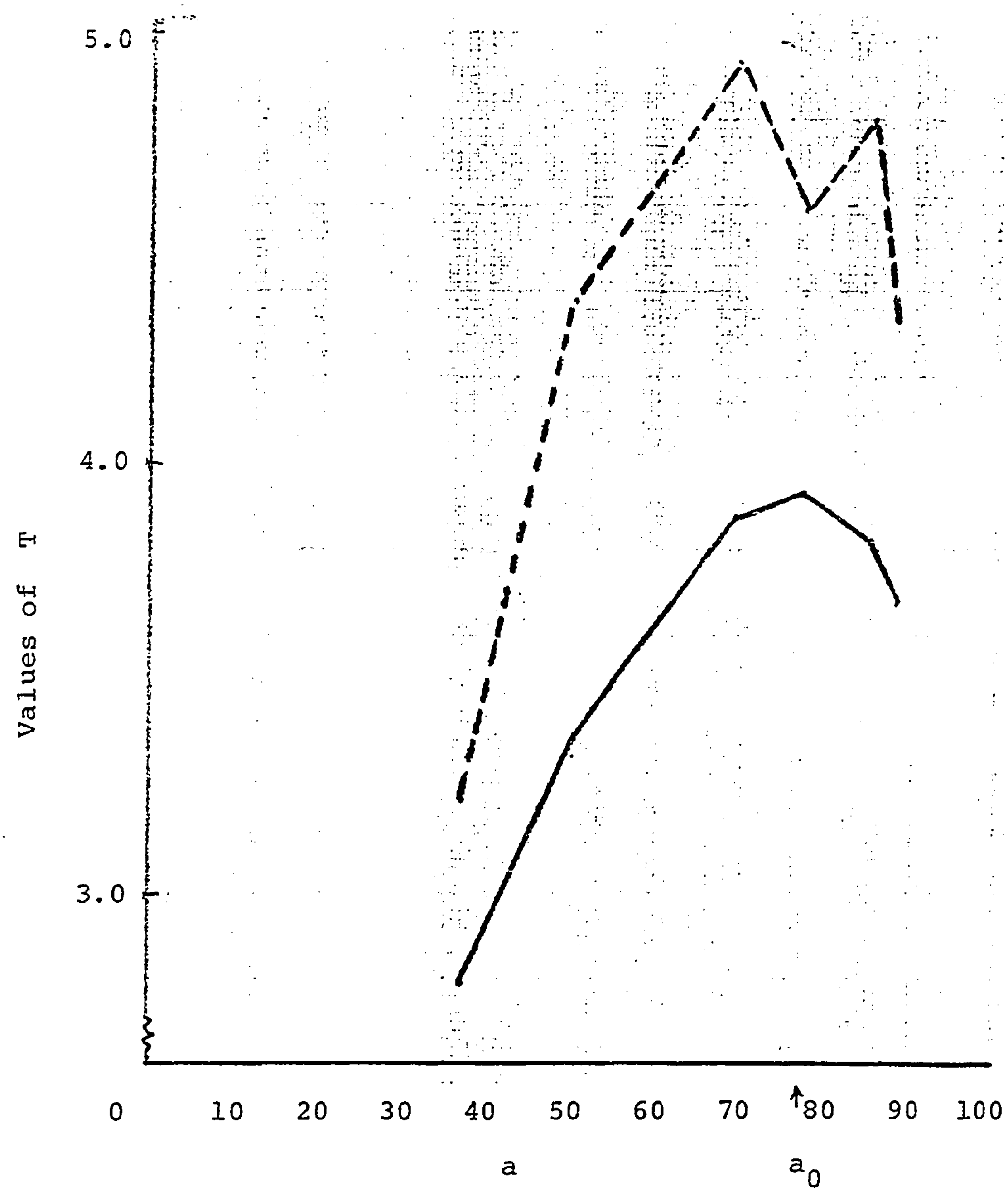


Figure 36. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when $\sigma^2_{xy}=.9$, $\sigma^2_e=.9$, and $\sigma^2_\varepsilon=.5$ $\underline{a_0}$ is expected optimum allocation

Table 32. Values of theoretical T, empirical T, empirical $\sqrt{F}$ and standard deviation of $\sqrt{F}$ at different allocations for the set of data with parameters $\sigma^2_{xy} = .9$, $\sigma^2_e = .9$, and $\sigma^2_\varepsilon = .9$

	Allocation	Theoretical T	Empirical T	Empirical $\sqrt{F}$	S.D. of $\sqrt{F}$	Sample
	35	2.14	2.34	32.25	4.46	1
	46	2.36	2.45(2.29)[a]	34.19(66.96)	3.99(4.85)	3,4[b]
	50	2.40	3.15	37.42	6.55	2
	51	2.40	2.72(2.49)	36.21(69.78)	7.19(6.97)	3,4
$a_0 =$	56	2.40	2.56(2.77)	32.76(73.42)	6.48(5.99)	2,4
				35.27	7.25	3
	62	2.40	2.47	33.18	3.79	2
	72	2.25	2.55	33.66	5.10	2
	75	2.18	2.58	33.89	6.03	1

[a]Values in parentheses are from 4.

[b]4 is a sample of size 40.

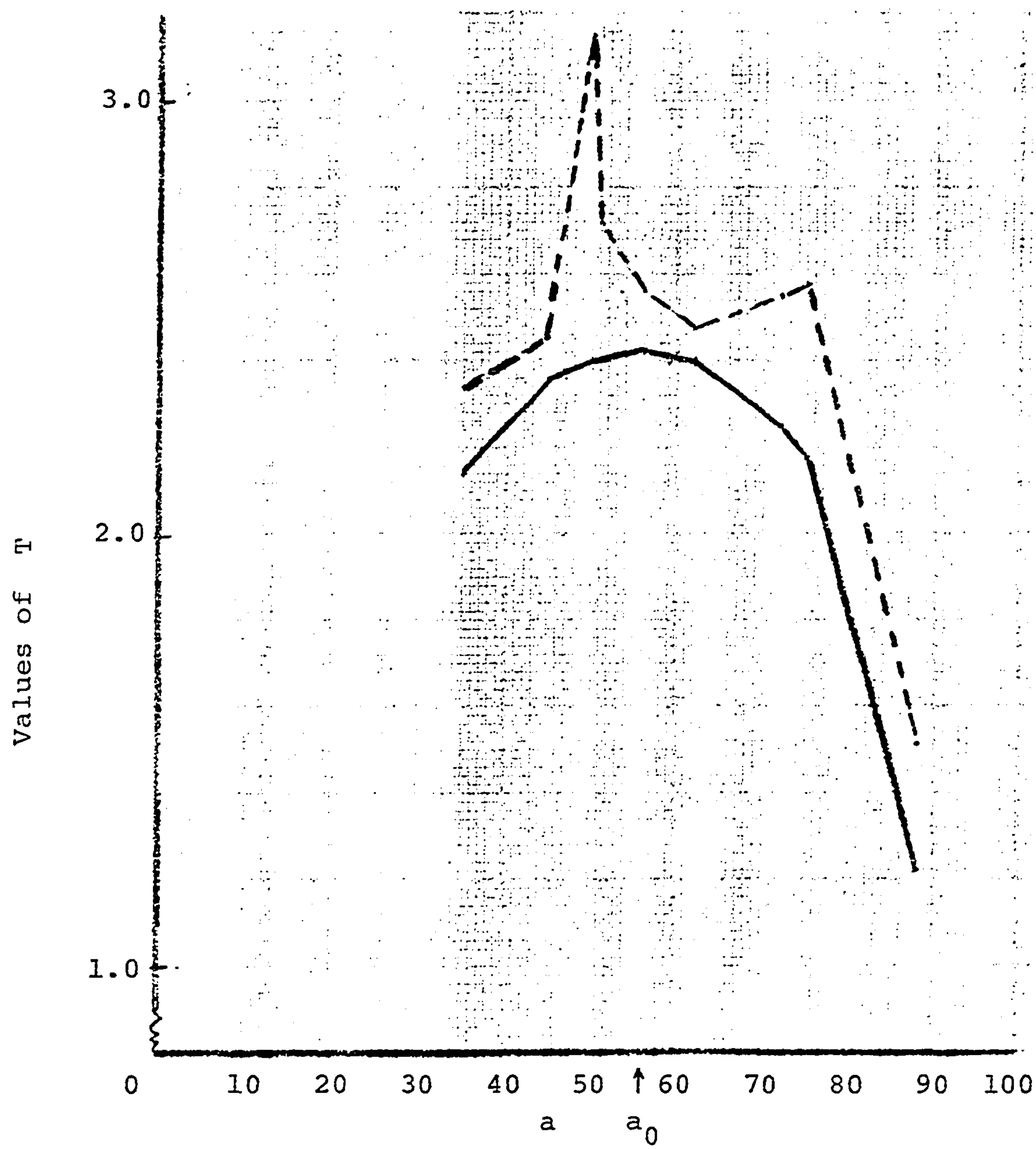


Figure 37. Empirical values (dashed line) and theoretical values (solid line) of T at different a's for the case when σ^2_{xy}=.9, σ^2_e=.9, and σ^2_ε=.9 $\underline{a_0}$ is expected optimum allocation

the latter F is larger than the former. Therefore, this transformation to T of the empirical F using its approximation to the former causes the differences; the values of the empirical T are higher than those of the theoretical T. This case, as mentioned earlier does not affect any interpretation of the utility of the strategy. The relationship or similarity of the theoretical and empirical values is the essence. After examining all the figures and tables, a summary of results of optimum allocation strategy for all sets of values is displayed in Table 33.

These twenty-seven results were evaluated according to two criteria, success and utility. Three levels of success were defined. If the Monte Carlo results peaked at the same level or nearly the same level as the theoretical results, this was called "successful," if the Monte Carlo results were lowest for two extreme values of $\underline{a}$ but the optimum Monte Carlo result was not at the theoretical optimum, this was termed "somewhat successful," if the theoretical result appeared independent of the Monte Carlo results, this was called "unsuccessful." Of course, practical constraints in the Monte Carlo results left considerable stochastic variability so that under certain circumstances both success and lack of success may have been due to chance.

Table 33. Summary of the results of the optimum allocation strategy in 27 sets of parameters

Set of parameters				Result of the optimum allocation strategy			Utility
No.	σ^2_{xy}	σ^2_e	σ^2_ε	Not Successful	Somewhat Successful	Successful	
1	1	1	1	x			No
2	1	1	5	x			No
3	1	1	9	x			No
4	1	5	1		x		No
5	1	5	5		x		No
6	1	5	9	x			No
7	1	9	1			x	No
8	1	9	5			x	No
9	1	9	9		x		No
10	5	1	1	x			No
11	5	1	5	x			No
12	5	1	9		x		No
13	5	5	1			x	No
14	5	5	5	x			No
15	5	5	9		x		No
16	5	9	1			x	No
17	5	9	5	x			No
18	5	9	9			x	No

Table 33. (Continued)

Set of parameters				Result of the optimum allocation strategy			Utility
No.	σ^2_{xy}	σ^2_e	σ^2_ε	Not Successful	Somewhat Successful	Successful	
19	9	1	1		x		No
20	9	1	5		x		Yes
21	9	1	9			x	Yes
22	9	5	1			x	Yes
23	9	5	5			x	Yes
24	9	5	9		x		Yes
25	9	9	1			x	Yes
26	9	9	5		x		Yes
27	9	9	9		x		Yes

The utility of the results is based on the derivative of <u>a</u> . For certain values of the parameters, most notably when σ^2_{xy} is large, the optimum is clearly defined; that is, the power of the test drops off sharply when one deviates from the optimum value for <u>a</u>. In this case optimum allocation is important and useful. When σ^2_{xy} is low, however, power is only slightly affected by failure to optimumally allocate; the power stays essentially the same for a broad range of <u>a</u> values.

Success and utility are not independent. If utility is low, then success, or lack of success, is probably a chance result. If utility is high, however, one might expect confirmation of the theoretical results.

The results from Figures 11 through 37, as summarized in Table 33, indicate that in every case the shape of the Monte Carlo result is similar to the theoretically derived expectation. If the curve is peaked, successful results are obtained. If the curve is flat, the Monte Carlo results are flat.

In terms of utility only σ^2_{xy} seems important. When σ^2_{xy} is high and error is at least moderate for X and/or Y, then optimum allocation seems useful and the Monte Carlo results are consistent with the derived ones. If σ^2_{xy} is low or there is little error to begin with, then there is little utility in optimum allocation.

Given high σ^2_{xy} and substantial error somewhere, then the relative size of σ^2_e and σ^2_ε, as well as their absolute values, determines the optimum values of a. The Monte Carlo and the derived results indicate that when $\sigma^2_e = \sigma^2_\varepsilon$, more resources should be allocated to the variate. Given this conclusion, the greater allocation goes to the variable with the larger error of the two.

It should be recognized that these results are based on the analysis of computer generated data. Although actual data were not used, an example illustrates how one could verify the asymptotic theoretical results using real data. Stroud (1972) provides a good example selected for this purpose. The expectation from the analysis of Stroud's data is shown in Figure 19, a flat curve with a low σ^2_e and σ^2_ε. The reason for selecting Stroud's example was that it fits best the derived optimum allocation formula:

(1) He used psychological tests, with 403 items of the Iowa Test of Educational Development as a covariate, X and 190 items of the Test of Academic Progress as a variate, Y,

(2) There are two groups, i.e., $i = 1,2$ for boys and girls,

(3) Sample sizes are large, 1974 girls and 1921 boys, and

(4) $\overline{X}_{boys} = \overline{X}_{girls}$. Table 34 was abridged from Table 2 of Stroud (1972) when ME variance of X and Y were scaled from 2.96 and 2.72 respectively to unity.

Table 34. Summary of data from Stroud (1972).

Group	Sample Size	$\bar{X}$	$\bar{Y}$	S_{XX}	S_{YY}	S_{XY}	F
Girls	1974	29.13	30.48	29.98	28.95	26.21	9.12***
Boys	1921	29.13	30.14	36.06	33.38	30.95	

***p < .001

Based on the above data, the following information was computed

$$r_{XX} = .97 \quad \text{or} \quad \sigma^2_\varepsilon = .03$$

$$r_{YY} = .97 \quad \text{or} \quad \sigma^2_e = .03$$

$$r^2_{XY} = .79 \quad \text{or} \quad \sigma^2_{xy} = .84$$

and $$nk^2_\alpha = 1.7053 \ .$$

The total number of testing items was 593. Applying the derived optimum allocation formula, the maximum power F test was expected at $a_0 = 307$. Using $a_0 = 307$ the empirical maximum F was computed from

$$F = 1 + \frac{nk^2_\alpha\left[\sigma^2_x + \dfrac{\sigma^2_\varepsilon}{286/403}\right]}{\left[\sigma^2_x + \dfrac{\sigma^2_\varepsilon}{286/403}\right]\left[\sigma^2_y + \dfrac{\sigma^2_e}{307/190}\right] - \sigma^2_{xy}} \ .$$

The F of 9.23 was the result, which is larger than the F of 9.12 of Stroud's for $\underline{a} = 190$. However, to determine the success of the optimum allocation strategy, values of F at other allocations were computed. Table 35 exhibits those values of F for different allocations.

Table 35. F Values at different allocations for Stroud's data

	Allocation	F Value
	100	8.72
	190	9.12
$a_0 =$	307	9.23
	400	9.03

It is clear from Table 35 that the maximum power F-test was obtained at the optimum allocation, $a_0 = 307$. With the allocation of smaller or greater than 307, less power was the result. However, as the theoretical curve for this case is flat, variation among values of F at different allocation is small. An F value of 9.23 is still highly significant at $p < .001$. While the optimum allocation is successful in terms of giving a maximum power, the improvement in the power or the utility is not large in this case. (From examining Figure 19, the curve is flat).

In conclusion, the results of the Monte Carlo investigation mostly confirm the expectation arising from the optimum allocation strategy. The strategy was successful and had some utility whenever the correlation between the variate and the covariate was high. This correlation is the most influential to the success and efficiency of the optimum allocation strategy. With a moderate correlation between the two variables of ANOCO, the strategy shows some success but little utility. In this case no significant improvement or increase in power of F-test was obtained. The flatness of the curves or the result from actual data were the evidence. The effect of ME of the variate was less influential than that of the correlation but, it is more influential than the effect of ME variance of the covariate. The greater importance of ME variance of the variate than that of covariate was demonstrated as Figure 10 indicates. With the two ME variance equal, never is allocation to the variate less successful than allocation to the covariate.

For much research in psychology and education, such as experiments on learning, split-plot analysis is competitive with ANOCO. Split-plot analysis with equal allocation of resources however, has been more often used than ANOCO. The design of a study might be as follows:

	Pretest	Treatment	Posttest
Group 1	Yes	Yes	Yes
Group 2	Yes	No	Yes

The observational units are randomly assigned to groups, the groups are next treated differently, and finally assessed on the posttest, the same as, or a comparable measure to the pretest. The person is considered the "whole plot" and the two measures of him are analogous to the "split-plot." Such analyses are referred as "repeated measure" designs in social science research.

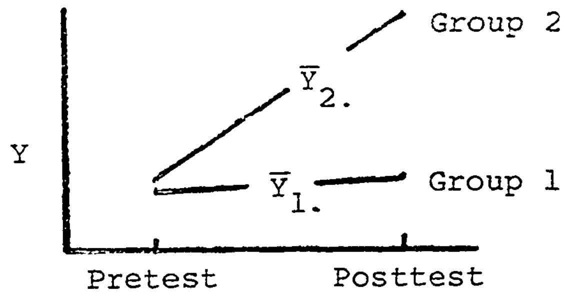
This design may be analyzed according to the model

$$Y_{ijk} = \mu + \alpha_i + \beta_{ij} + \gamma_k + \alpha\gamma_{ik} + \ell_{ijk}$$

$$\ell_{ijk} \sim NID(0, \sigma_\ell^2)$$

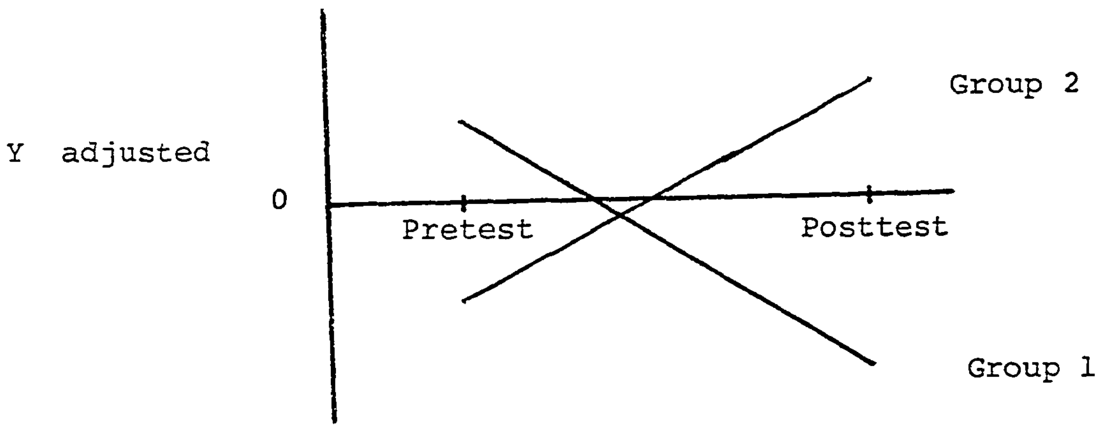
$$\beta_{ij} \sim NID(0, \sigma_\beta^2) \quad .$$

We would like to convince the reader that ANOCO is superior to the "split-type" analyses for this design. That part of the data most relevant for the assessment of treatment differences is the posttest results. Because of the randomization the experimenter expects no pretest differences. The pretest however, is expected to reduce the error by allowing control for individual differences within groups. A typical case might appear as follows:



The analyses of the split plot results in two significance tests sensitive to these expected results. The main plot test is sensitive to the difference between groups on the mean of both measures (i.e., $\overline{Y}_{1.}-\overline{Y}_{2.}$). However these means should not be computed because it is not expected that $\overline{Y}_{11}-\overline{Y}_{12} = \overline{Y}_{21}-\overline{Y}_{22}$. That is, under non-null conditions, the groups would differ only with respect to the posttest.

The sub-plot test is sensitive to differences between the groups on pretest vs. postest differences. That is, this test takes cognizance of residuals after pretest and group differences are adjusted. It tests that $\overline{Y}_{11}-\overline{Y}_{12} = \overline{Y}_{21}-\overline{Y}_{22}$. These adjusted means might appear as follows:



Given the results depicted in the first figure, both tests are somewhat sensitive to these differences. Therefore, it follows that neither test is optimally sensitive to these differences. On the other hand ANOCO emphasizes posttest differences while adjusting for chance differences which may have occurred on the pretest. This analysis directs its sensitiveness to the expected non-null conditions.

Also, in order to meet the homoscedastic assumption, one must allocate resources equally to the pre and post measures for the split-plot analysis. As has been shown herein, if a unit of measurement is the same for X and Y, in ANOCO one should invest more resources in the Y measure.

Generally, one concludes the ANOCO procedure is substantially better by virtue of the analysis' directly assessing expected differences and providing for a more efficient expenditure of resources than ANOVA.

SUMMARY

The purpose of this dissertation was to set up a strategy in allocation of measurement resources to obtain maximum power in ANOCO using OSM. Allocation for X and Y was intended to reduce the ME variance of X and Y given the fixed cost for sample size.

The steps of development started with the computation of the ordinary F-ratio based on OSM. With $\underline{a}$, the allocation to Y and $\underline{c-a}$, the allocation to X, the ME variances of Y and X were reduced by factors of $\frac{1}{a}$ and $\frac{1}{c-a}$ respectively. The differentiation of this F-ratio with respect to $\underline{a}$, for fixed c, led to the final formula for the optimum allocation,

$$a_0 = \frac{c + \dfrac{\sigma_\varepsilon^2}{1-\sigma_\varepsilon^2}}{1 \pm \sigma_{xy} \dfrac{\sigma_\varepsilon}{(1-\sigma_\varepsilon^2)^{1/2}} \dfrac{(1-\sigma_e^2)^{1/2}}{\sigma_e}}$$

where σ_{xy} = the true correlation between X and Y,

σ_ε^2 = the ME variance of X

and σ_e^2 = the ME variance of Y.

This formula suggests that the more ME variance of the variate, the more the allocation to the variate. With smaller σ_{xy}^2, the more the allocation of resources should go to Y as

compared to X. With $\sigma_{xy}^2 = 0$, no allocation of measurement resource at all is required for X.

The Monte Carlo investigation of some 27 sets of values for different combination of the parameters σ_e^2, σ_ε^2, and σ_{xy}^2 compared empirical values with theoretical expectations of efficiency. The comparison suggested the success of the strategy for those sets of values with rather high σ_{xy}^2 and/or high σ_e^2. With very low σ_{xy}^2 and low σ_e^2, no matter what σ_ε^2 was, no successful application of the strategy was obtained. In terms of both utility and efficiency, the strategy was successful only for the mentioned cases with high σ_{xy}^2 (.9). The reason for the lack of practical utility in the other cases was a very flat theoretical allocation curve; any reasonable allocation resulted in practically the same power as the optimum one.

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